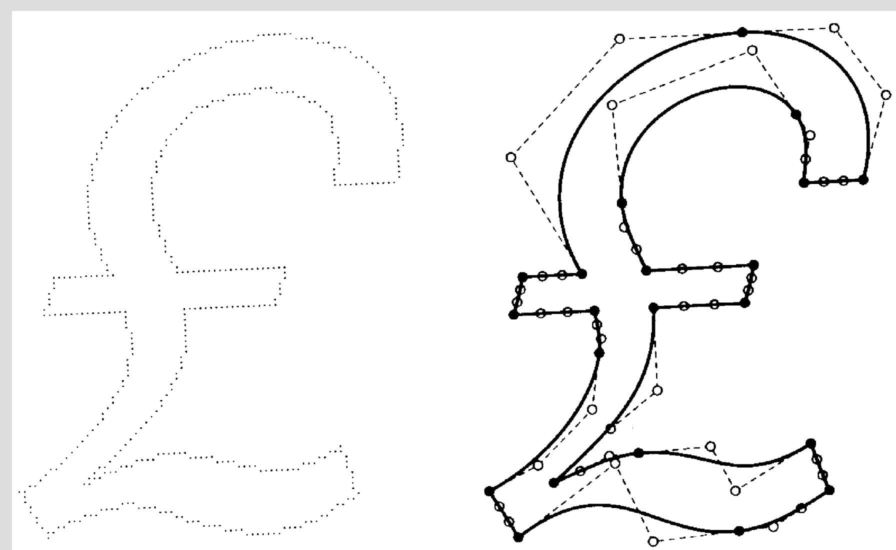
Bezier Curves, B-Splines, NURBS

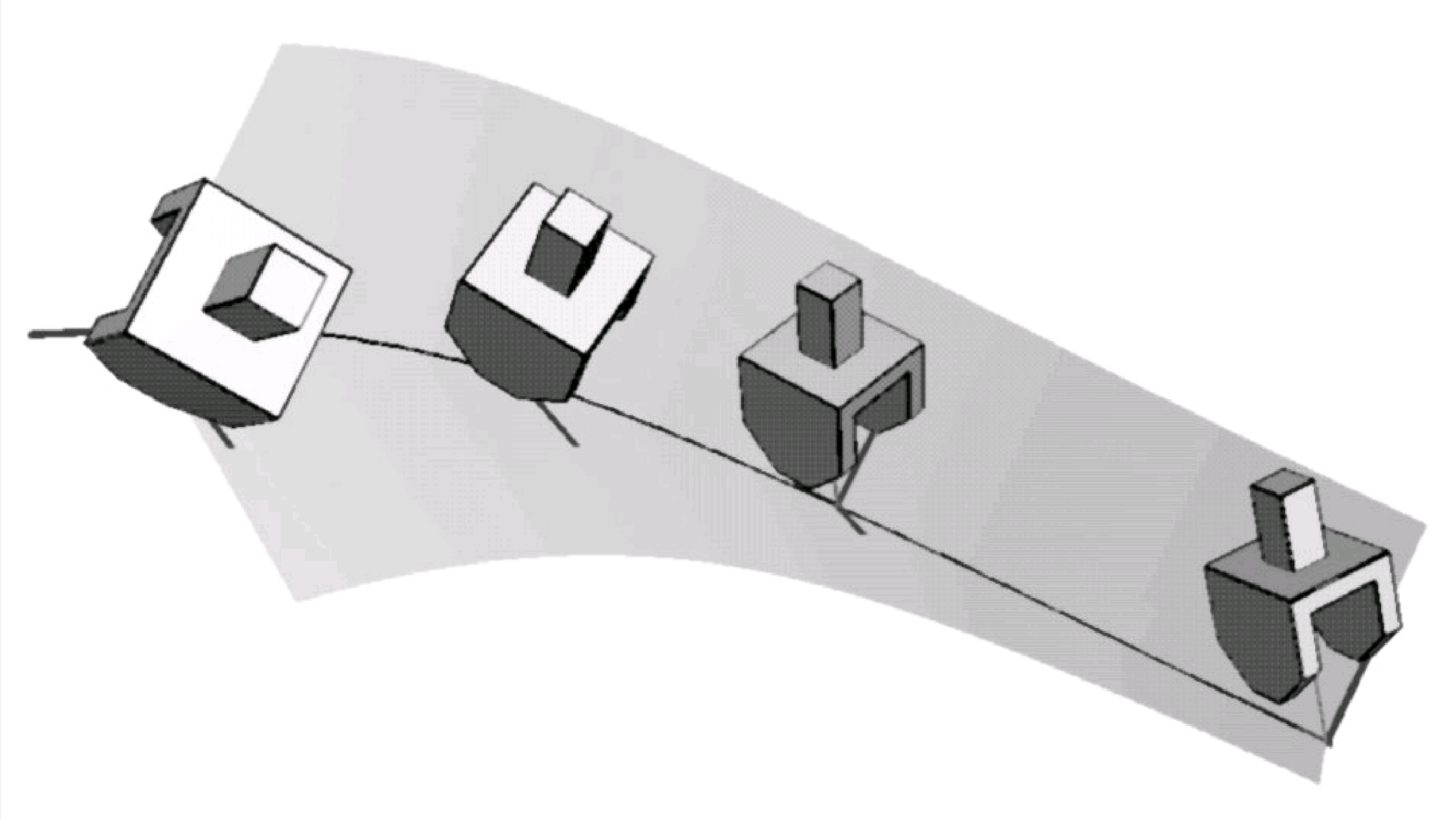
Example Application: Font Design and Display

- Curved objects are everywhere
- There is always need for:
  - mathematical fidelity
  - high precision
  - artistic freedom and flexibility
  - physical realism



Example Application: Graphic Design and Arts

Example Application: Tool Path Generation and Motion Planning



Functional Representations

- ***Explicit Functions***:
  - representing one variable with another
  - fine if $\exists$ only one x value for each y value
  - Problem: what if I have a sphere?

$$z = \sqrt{r^2 - x^2 - y^2}$$

- Multiple values …. (not used in graphics)

Functional Representations

- ***Implicit Functions:***
  - curves/surfaces represented as “the zeros”
  - good for rep. of *n-1*-D objects in *n*-D space
  - Sphere example:

$$x^2 + y^2 + z^2 - r^2 = 0$$

  - What class of function?
    - *polynomial*: linear combo of integer powers of x,y,z
    - *algebraic curves & surfaces:* rep’d by implicit polynomial functions
    - *polynomial degree:* total sum of powers,
      i.e. polynomial of degree 6:

$$x^2 + y^2 + z^2 - r^2 = 0$$

Functional Representations

- ***Parametric Functions:***
  - 2D/3D curve: two functions of one parameter
    (x(u), y(u)) (x(u), y(u), z(u))
  - 3D surface: three functions of two parameters
    (x(u,v), y(u,v), z(u,v))
  - Example: Sphere
    Note: rep. not algebraic, but is parametric

$$
\begin{aligned}
x(\theta, \phi) &= \cos\phi\cos\theta \\
y(\theta, \phi) &= \cos\phi\sin\theta \\
z(\theta, \phi) &= \sin\phi
\end{aligned}
$$

Functional Representations

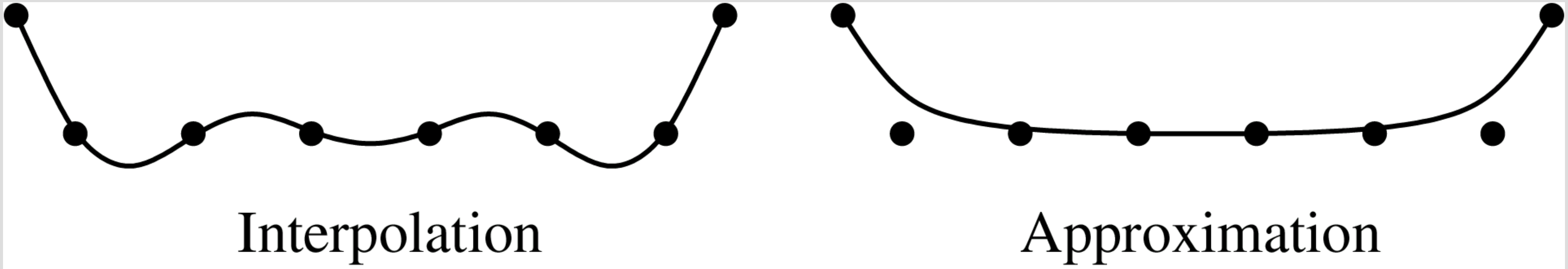
- Which is best??
  - It depends on the application
  - Implicit is good for
    - computing ray/surface intersection
    - point inclusion (inside/outside test)
    - mass & volume properties
  - Parametric is good for
    - subdivision, faceting for rendering
    - Surface & area properties
    - popular in graphics

Issues in Specifying/Designing Curves/Surfaces

- Note: the internal mathematical representation can be very complex
  - high degree polynomials
  - hard to see how parameters relate to shape
- How do we deal with this complexity?
  - Use *curve control points* and either
    - Interpolate
    - Approximate

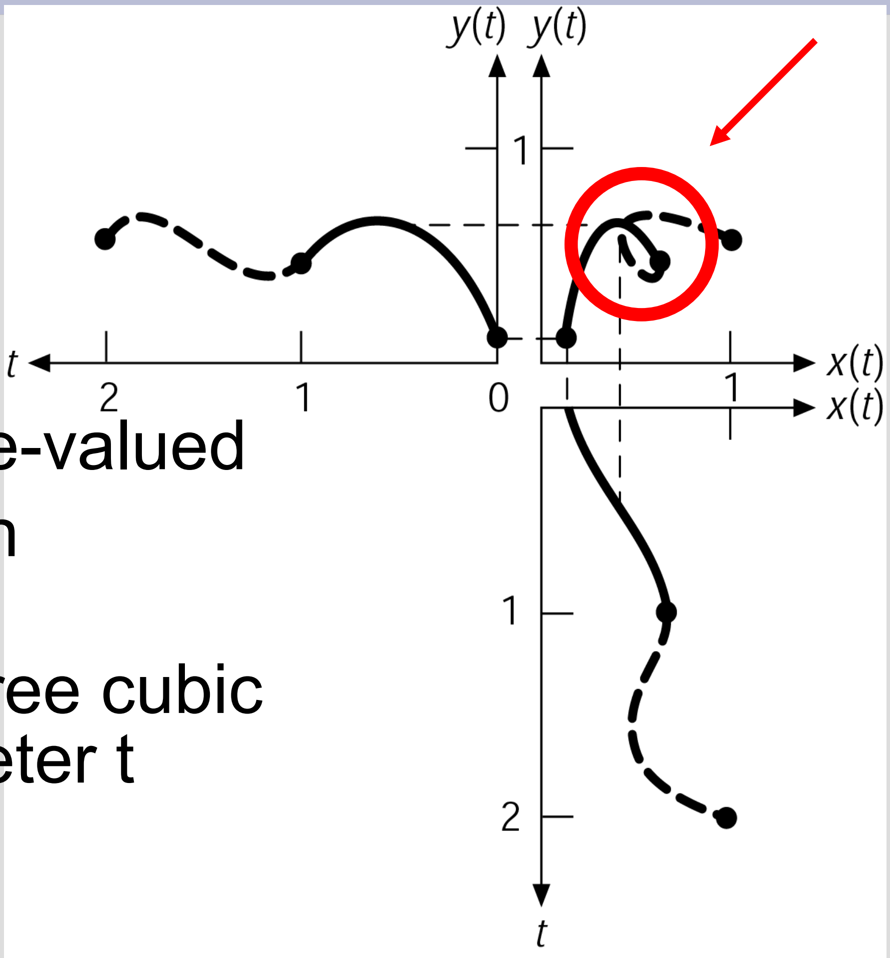
Points to Curves

- The *Lagrangian interpolating polynomial*
  - $n+1$ points, the unique polynomial of degree n
  - curve wiggles thru each control point
  - Issue: not good if you want smooth or flat curves
- *Approximation* of control points
  - points are *weights* that tug on the curve or surface



Parametric Curves

- General rep:
  $$x = x(t),\ y = y(t),\ z = z(t)$$
- Properties:
  - individual functions are single-valued
  - approximations are done with piecewise poly curves
  - Each segment is given by three cubic polynomials (x,y,z) in parameter t
  - Concise representation



Cubic Parametric Curves

- Balance between
  - Complexity
  - Control
  - Wiggles
  - Amount of computation
  - Non-planar

Parametric Curves

- Cubic Polynomials that define a parametric curve segment

$$Q(t) = [x(t)\ y(t)\ z(t)]^T$$

are of the form

- Notice we restrict the parameter t to be

$$0 \le t \le 1.$$

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x,$$
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y,$$
$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z,$$
$$0 \le t \le 1.$$

Parametric Curves

- If ***coefficients*** are represented as a matrix

a

$$C = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix}$$

tl

$$T = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}^T$$

$$Q(t) = [x(t)\ t(t)\ z(t)]^T = C \cdot T$$

Parametric Curves

- $Q(t)$ can be defined with four constraints
  - Rewrite the coefficient matrix C as $C = G \cdot M$ where M is a 4x4 ***basis matrix***, and G is a four-element constraint matrix (***geometry matrix***)
- Expanding $Q(t) = G \cdot M \cdot T$ gives:

$$Q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} G_1 & G_2 & G_3 & G_4 \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$Q(t)$ is a weighted sum of the columns of the geometry matrix, each of which represents a point or vector in 3-space

Parametric Curves

- Multiplying out $x(t) = G_x \cdot M \cdot T$ gives

$$x(t) = (t^3 m_{11} + t^2 m_{21} + t m_{31} + m_{41}) g_{1x} + (t^3 m_{12} + t^2 m_{22} + t m_{32} + m_{42}) g_{2x}$$
$$+ (t^3 m_{13} + t^2 m_{23} + t m_{33} + m_{43}) g_{3x} + (t^3 m_{14} + t^2 m_{24} + t m_{34} + m_{44}) g_{4x}$$

  (i.e. just weighted sums of the elements)

- The weights are cubic polynomials in t (called the ***blending functions***, $B=MT$)
- M and G matrices vary by curve
  - Hermite, Bézier, spline, etc.

Warning, Warning, Warning: Pending Notation Abuse

- *t* and *u* are used <u>interchangeably</u> as a parameterization variable for functions
- Why?
  - *t* historically is "time", certain parametric functions can describe "change over time" (e.g. motion of a camera, physics models)
  - *u* comes from the 3D world, i.e. where two variables describe a B-spline surface
    - *u* and *v* are the variables for defining a surface
- Choice of *t* or *u* depends on the text/reference

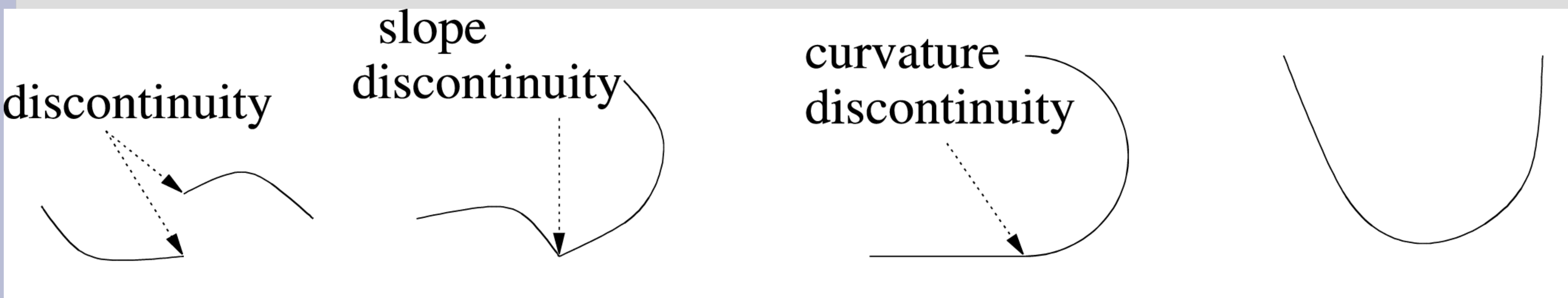
Continuity

Two types:

- Geometric Continuity, G^i:
  - endpoints meet
  - tangent vectors' directions are equal
- Parametric Continuity, C^i:
  - endpoints meet
  - tangent vectors' directions are equal
  - tangent vectors' magnitudes are equal
- In general: C implies G but not vice versa

Parametric Continuity

- ***Continuity*** (recall from the calculus):
  - Two curves are C^i continuous at a point p iff the i-th derivatives of the curves are equal at p



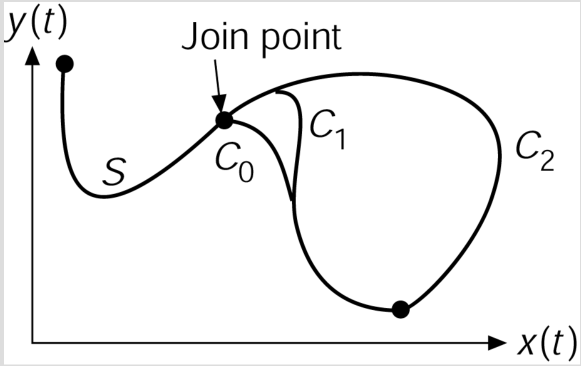
Not continuous C^0 continuous C^1 continuous C^2 continuous

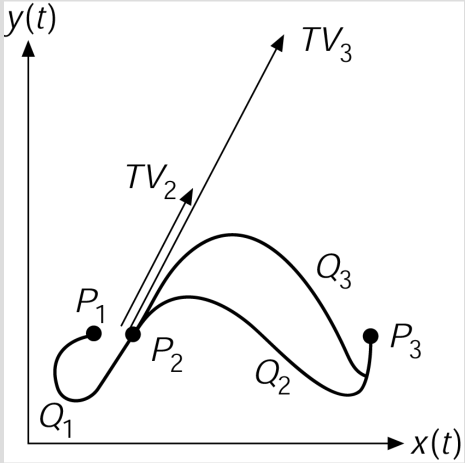
Continuity

- The derivative of $Q(t)$ is the parametric tangent vector of the curve:

$$\frac{d}{dt}Q(t) = Q'(t) = \left[\begin{array}{ccc} \frac{d}{dt}x(t) & \frac{d}{dt}y(t) & \frac{d}{dt}z(t) \end{array}\right]^T = \frac{d}{dt}C \cdot T = C \cdot \left[\begin{array}{cccc} 3t^2 & 2t & 1 & 0 \end{array}\right]^T =$$

$$\left[\begin{array}{ccc} 3a_x t^2 + 2b_x t + c_x & 3a_y t^2 + 2b_y t + c_y & 3a_z t^2 + 2b_z t + c_z \end{array}\right]^T$$

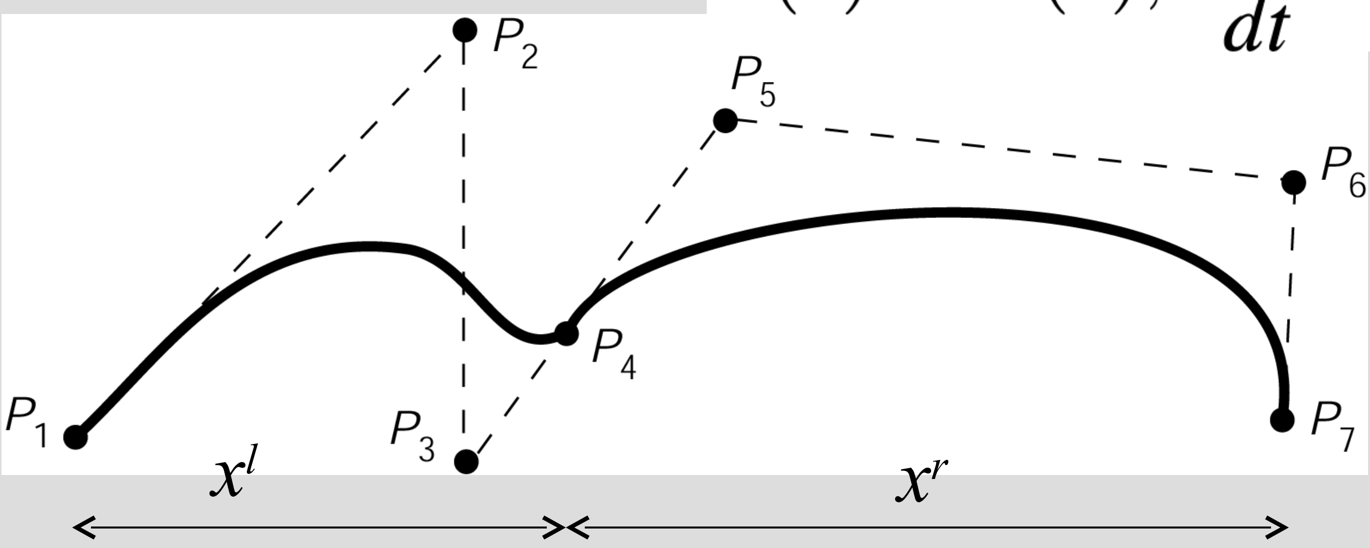




Continuity

- What are the conditions for C^0 and C^1 continuity at the joint of curves x^l and x^r?
  - tangent vectors at end points equal
  - end points equal

$$x^l(1) = x^r(0), \quad \frac{d}{dt}x^l(1) = \frac{d}{dt}x^r(0)$$

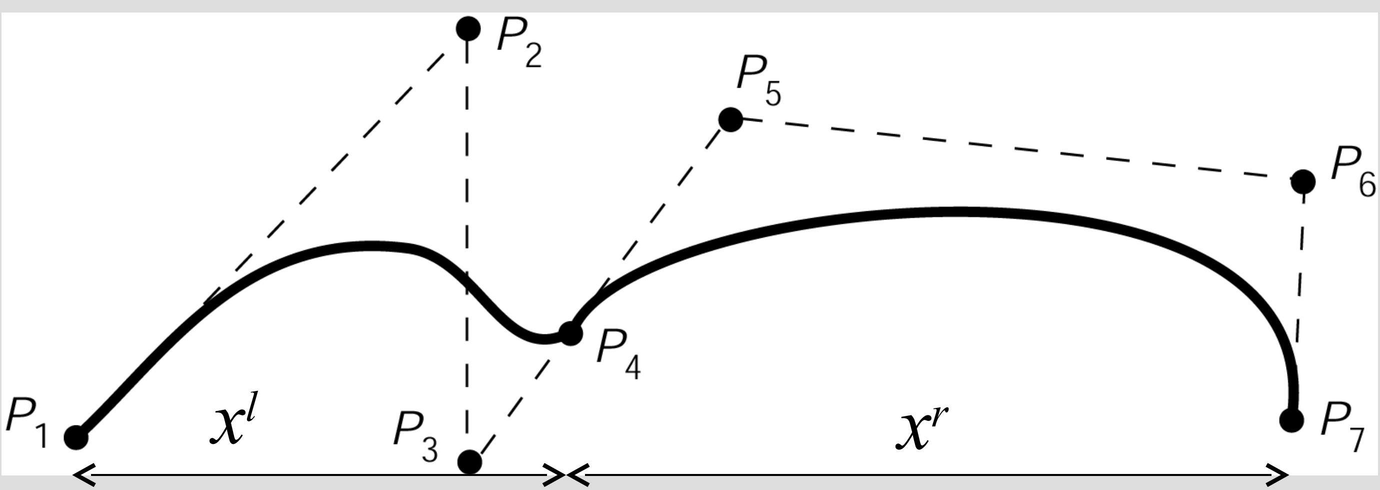


Continuity

- In 3D, compute this for each component of the parametric function
  - For the x component:

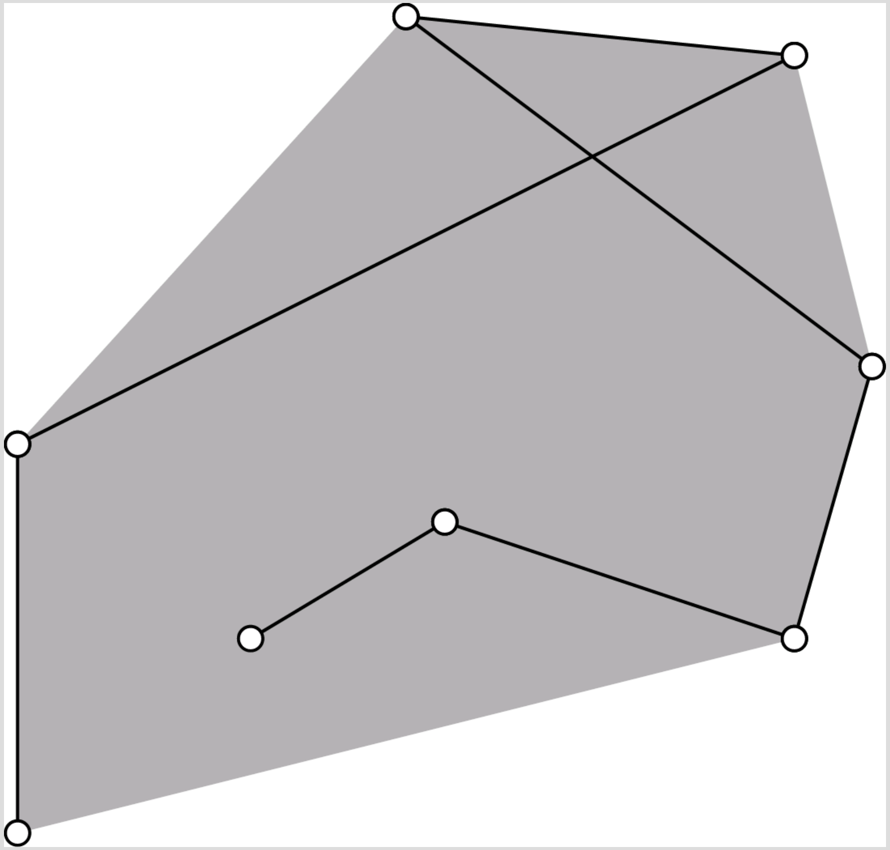
$$x^l(1) = x^r(0) = P_{4_x},\ \frac{d}{dt}x^l(1) = 3(P_{4_x} - P_{3_x}),\ \frac{d}{dt}x^r(0) = 3(P_{5_x} - P_{4_x})$$

- Similar for the y and z components.



Convex Hulls

- The smallest convex container of a set of points
- Both practically and theoretically useful in a number of applications

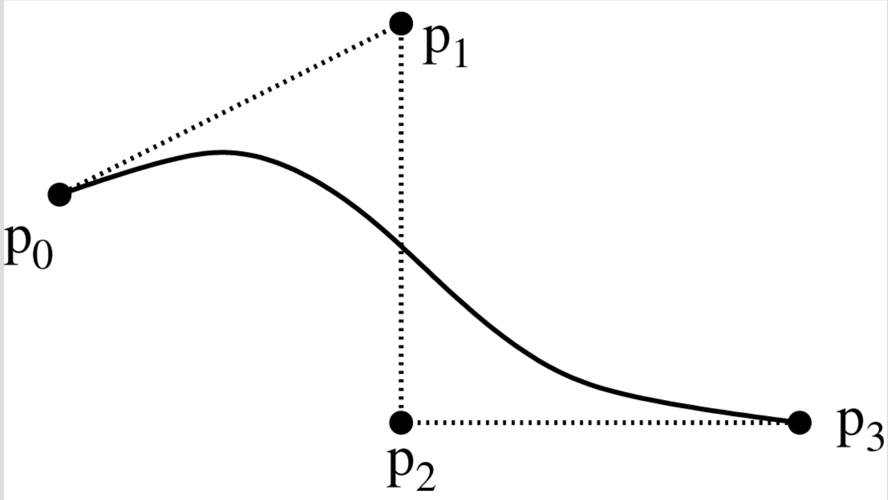


Some Types of Curves

- Hermite
  - def'd by two end points and two tangent vectors
- **Bézier**
  - two end points plus two control points for the tangent vectors
- **Splines**
  - <u>B</u>asis <u>Splines</u>
  - def'd w/ 4 control points
  - Uniform, nonrational *B-splines*
  - Nonuniform, nonrational B-splines
  - Nonuniform, rational *B-splines (**NURBS**)*

Bézier Curves

- Pierre Bézier @ Rénault ~1960
- Basic idea
  - four points
  - Start point $\boldsymbol{P_0}$
  - End point $\boldsymbol{P_3}$
  - Tangent at $\boldsymbol{P_0}$, $\underline{\boldsymbol{P_0 P_1}}$
  - Tangent at $\boldsymbol{P_3}$, $\underline{\boldsymbol{P_3 P_2}}$



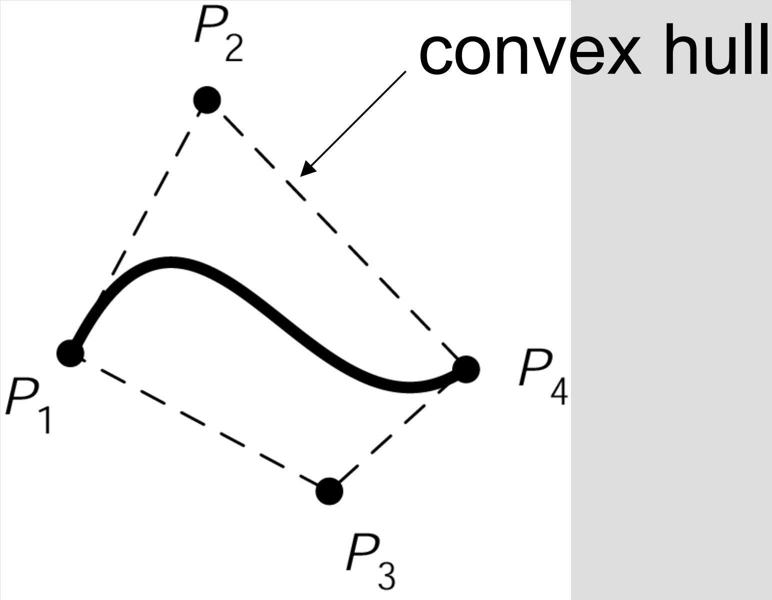
Bézier Curves

An Example:

- ***Geometry matrix*** is

$$G_B = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix}$$

  where P_i are control points for the curve
- ***Basis Matrix*** is



$$M_B = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Bézier Curves

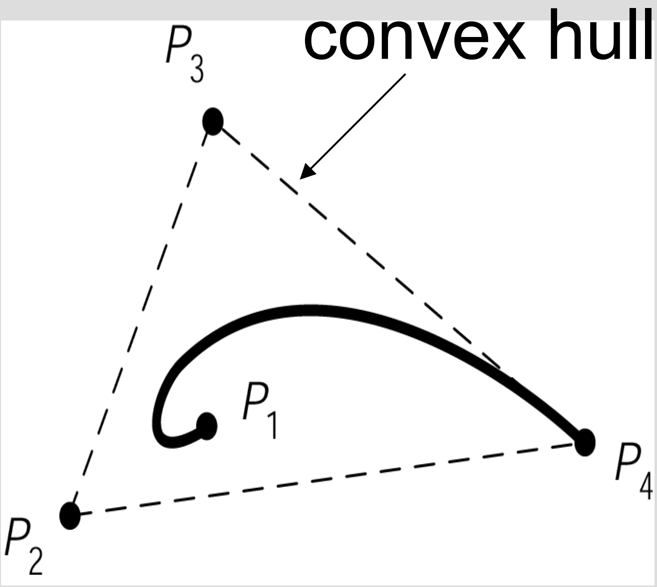
- The general representation of a Bézier curve is

  where $Q(t) = G_B \cdot M_B \cdot T$

  G_B - Bézier Geometry Matrix

  M_B - Bézier Basis Matrix

  which is (multiplying out):



$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

Bernstein Polynomials

- The general form for the *i*-th Bernstein polynomial for a degree *k* Bézier curve is

$$b_{ik}(u) = \binom{k}{i}(1-u)^{k-i}u^i.$$

- Some properties of BPs
  - Invariant under transformations
  - Form a *partition of unity*, i.e. summing to 1
  - Low degree BPs can be written as high degree BPs
  - BP derivatives are linear combo of BPs
  - Form a basis for space of polynomials w/ deg≤*k*

General Bezier Curve

$$s(t) = \sum_{i=0}^{n} p_i B_{n,i}(t)$$

Bernstein basis

$$B_{n,i}(t) = \binom{n}{i} t^i (1 - t)^{n-i}$$

The Quadratic and Cubic Curves of Java 2D are Bezier Curves with n=2 and n=3

The p_i are the control points

Bernstein Polynomials

- For those that forget combinatorics

$$b_{ik}(u) = \frac{k!}{i!(k-i)!}(1-u)^{k-i}u^{i}$$

Joining Bézier Segments: The Bernstein Polynomials

- Observe

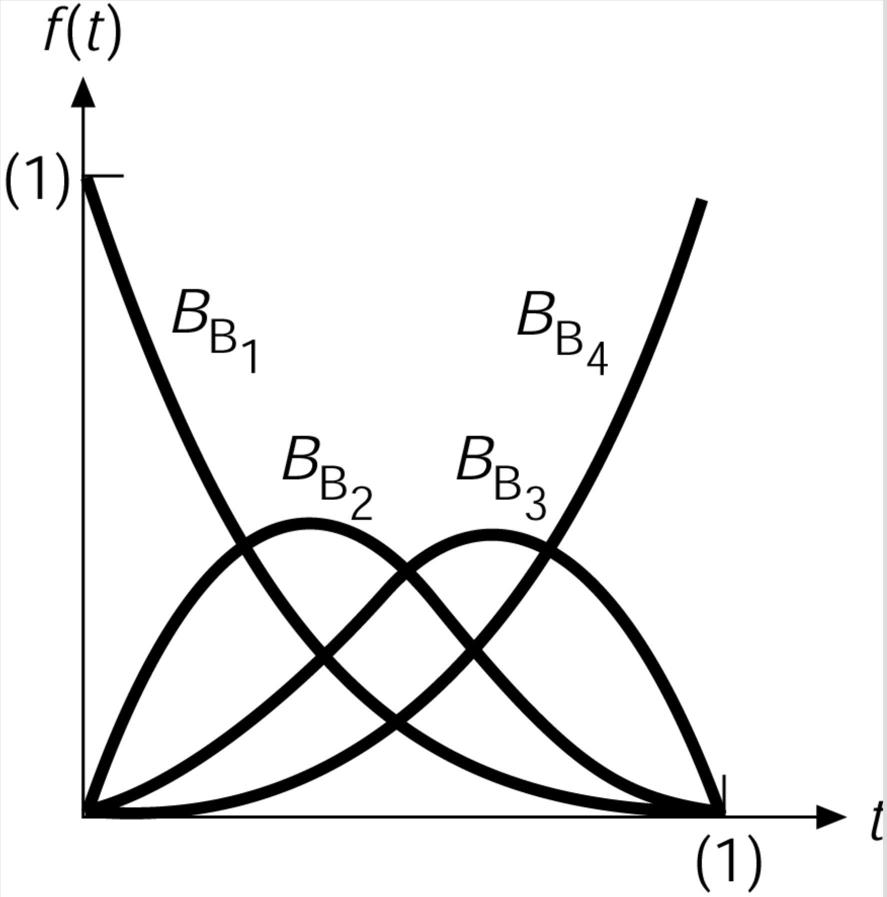
$$Q(t) = \underline{(1-t)^3} P_1 + \underline{3t(1-t)^2} P_2 + \underline{3t^2(1-t)} P_3 + \underline{t^3} P_4$$

  The Four *Bernstein polynomials*
  - also defined by

$$B_B = M_B \cdot T$$

- These represent the blending proportions among the control points

Joining Bézier Segments: The Bernstein Polynomials



- The four cubic *Bernstein polynomials*

$$B_B = M_B \cdot T$$

- Observe:
  - at t=0, only $\boldsymbol{B_{B1}}$ is >0
    - curve interpolates P1
  - at t=1, only $\boldsymbol{B_{B4}}$ is >0
    - curve interpolates P4

Joining Bézier Segments: The Bernstein Polynomials

- Cubic Bernstein blending functions

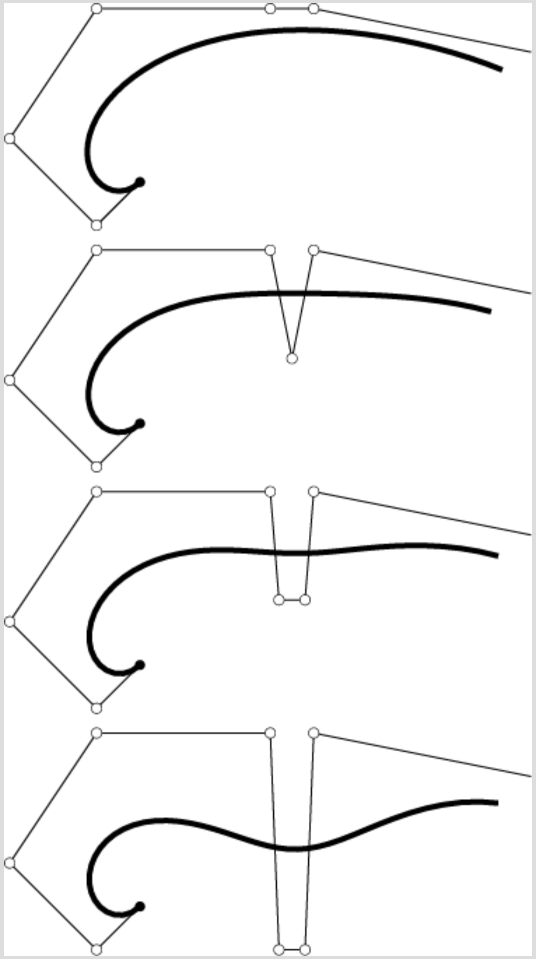
- Observe: the coefficients are just rows in Pascal's triangle

$$
\begin{aligned}
b_{03}(u) &= (1-u)^3 \\
b_{13}(u) &= 3u(1-u)^2 \\
b_{23}(u) &= 3u^2(1-u) \\
b_{33}(u) &= u^3.
\end{aligned}
$$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Properties of Bézier Curves

- Affine invariance
- Invariance under affine parameter transformations
- Convex hull property
  - curve lies completely within original control polygon
- Endpoint interpolation
- Intuitive for design
  - curve mimics the control polygon



Issues with Bézier Curves

- Creating complex curves may (with lots of wiggles) requires many control points
  - potentially a very high-degree polynomial
- Bézier blending functions have *global support* over the whole curve
  - move just one point, change whole curve
- *Improved Idea: link (C^1) lots of low degree (cubic) Bézier curves end-to-end*

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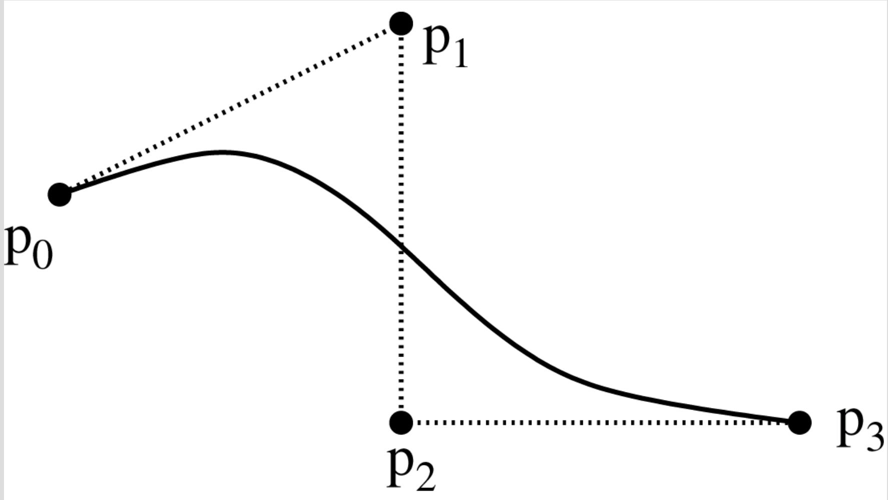

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$$\begin{aligned} b_{03}(u) &= (1-u)^3 \\ b_{13}(u) &= 3u(1-u)^2 \\ b_{23}(u) &= 3u^2(1-u) \\ b_{33}(u) &= u^3. \end{aligned}$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

B-Spline Curve

$$p(t) = \sum_{i=0}^{n} p_i N_{k,i}(t)$$

Defined only on $[t_3, t_{n+k-2})$

Normalized B-spline blending functions

$$N_{0,i}(t) = \begin{cases} 1, & t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

$$N_{k,i}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{k-1,i}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{k-1,i+1}(t)$$

n+1 control points and n+k+2 parameters known as knots

B-Spline to Bezier Conversion

If the knots are uniformly distributed

$$b_{-1} = (p_{i-1} + 2p_i)/3$$

$$b_1 = (2p_i + p_{i+1})/3$$

$$b_0 = (b_{-1} + b_1)/2$$

$$b_2 = (p_i + 2p_{i+1})/3$$

$$b_4 = (2p_{i+1} + p_{i+2})/3$$

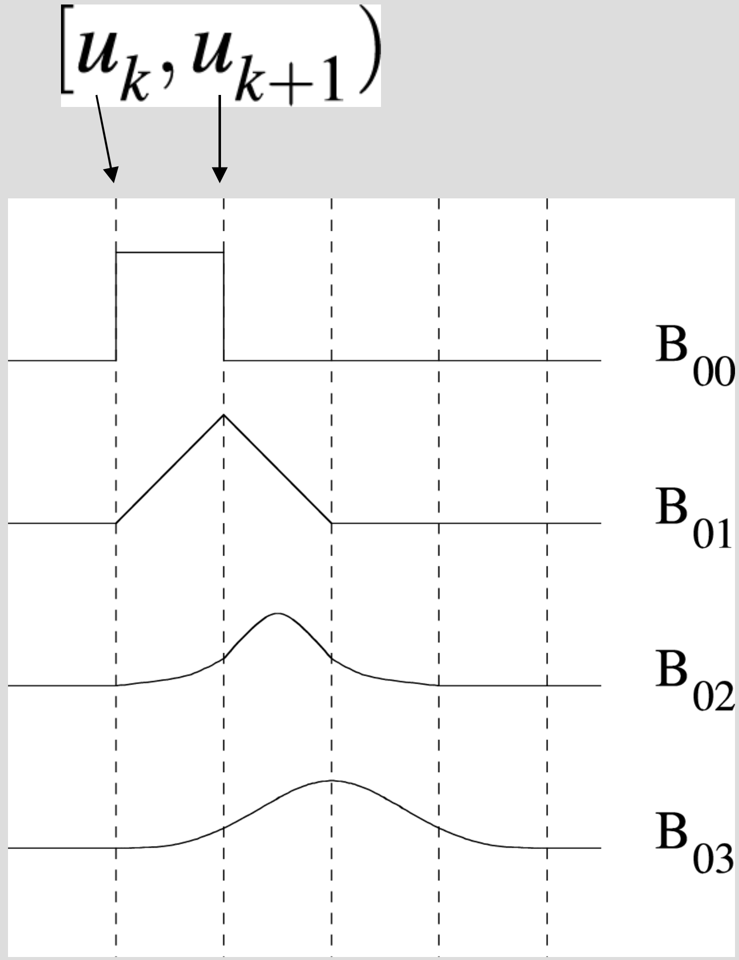
$$b_3 = (b_2 + b_4)/2$$

B-splines: Basic Ideas

- Similar to Bézier curves
  - Smooth blending function times control points
- But:
  - Blending functions are non-zero over only a small part of the parameter range (giving us *local support*)
  - When nonzero, they are the “concatenation” of smooth polynomials

B-spline Blending Functions

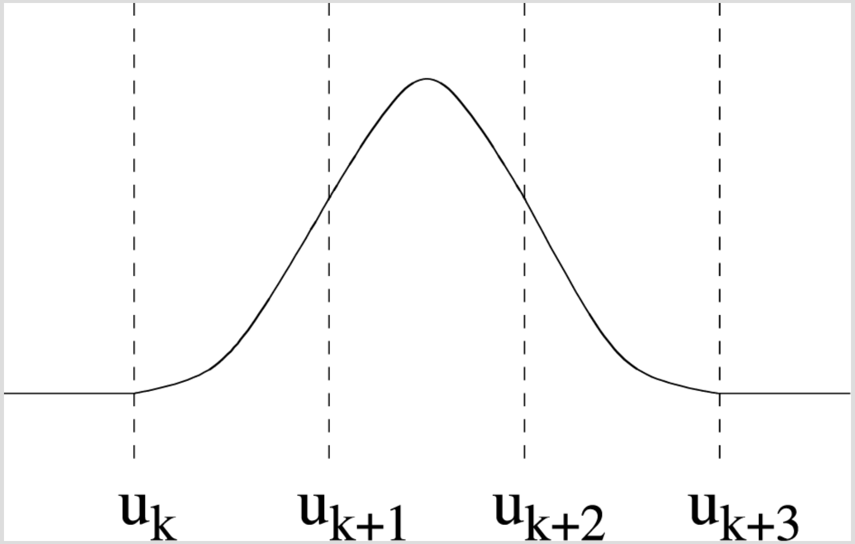
- $B_{k,0}(t)$ is a step function that is 1 in the interval
- $B_{k,1}(t)$ spans two intervals and is a piecewise linear function that goes from 0 to 1 (and back)
- $B_{k,2}(t)$ spans three intervals and is a piecewise quadratic that grows from 0 to 1/4, then up to 3/4 in the middle of the second interval, back to 1/4, and back to 0
- $B_{k,3}(t)$ is a cubic that spans four intervals growing from 0 to 1/6 to 2/3, then back to 1/6 and to 0

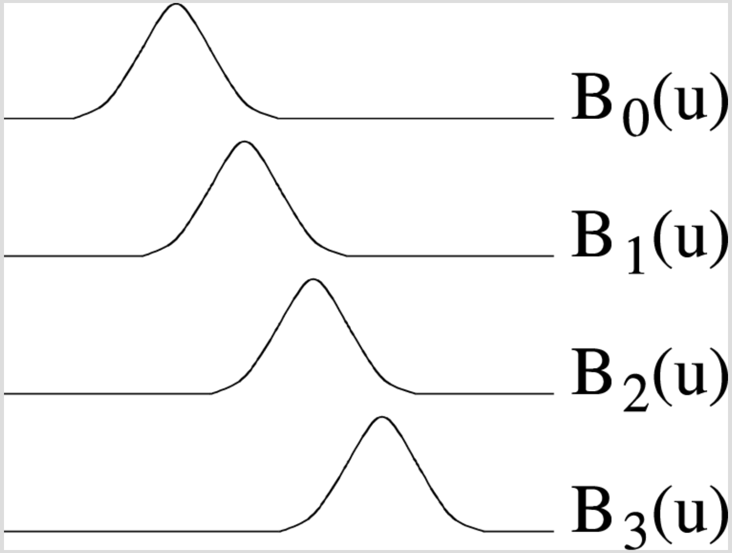


B-spline blending functions

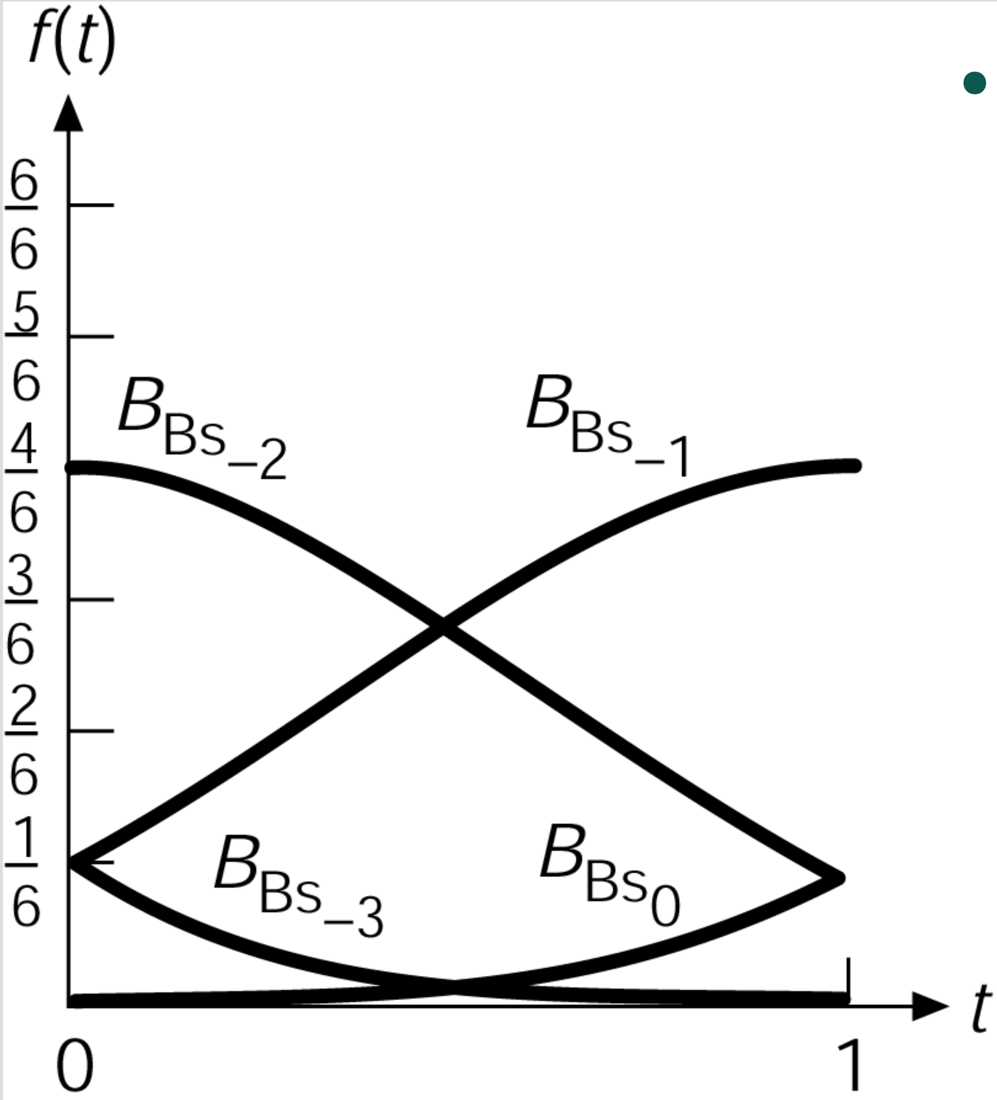
B-spline Blending Functions: Example for 2nd Order Splines

- Note: can't define a polynomial with these properties (both 0 and non-zero for ranges)
- Idea: subdivide the parameter space into *intervals* and build a *piecewise polynomial*
  - Each interval gets different polynomial function





B-spline Blending Functions: Example for 3d Order Splines

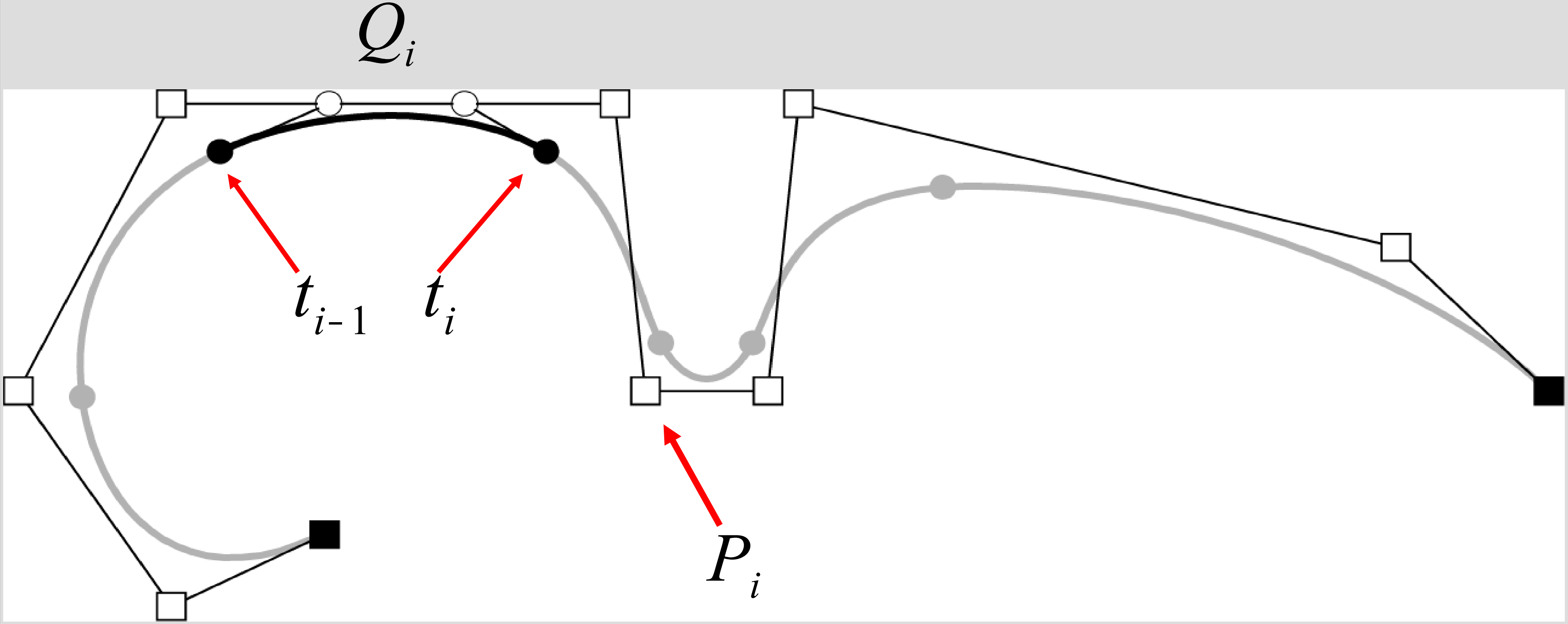


- Observe:
  - at t=0 and t=1 just three of the functions are non-zero
  - all are >=0 and sum to 1, hence the convex hull property holds for each curve segment of a B-spline

B-splines: Setting the Options

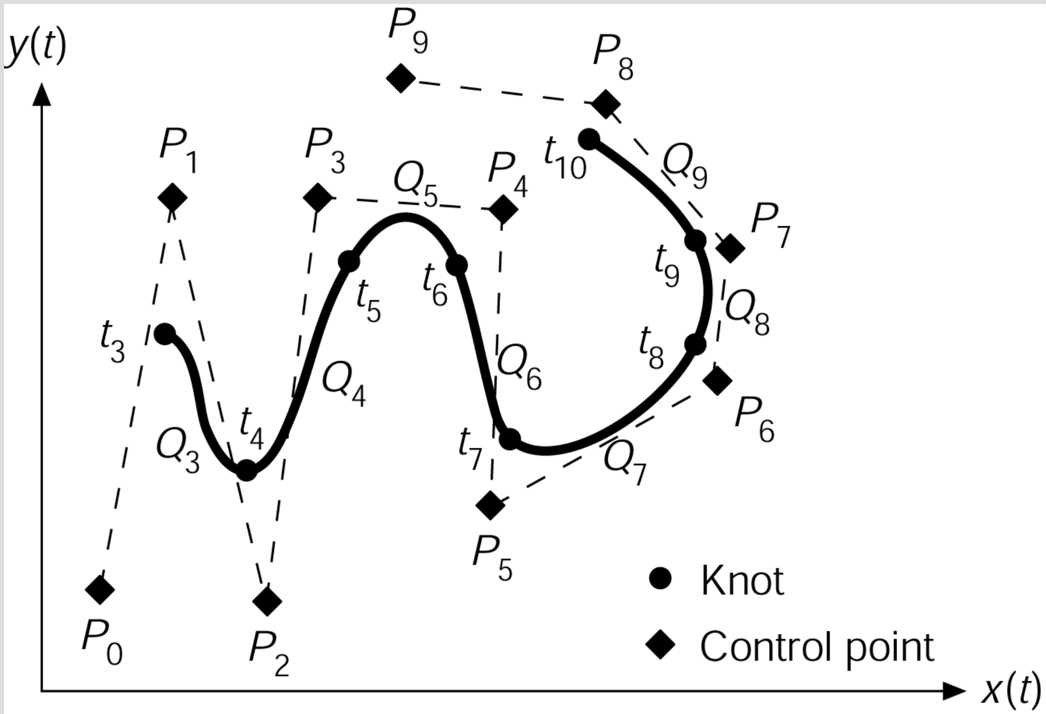
- Specified by
  - $m \geq 3$
  - *m*+1 ***control points***, $P_0 \ldots P_m$
  - *m*-2 cubic polynomial curve segments, $Q_3 \ldots Q_m$
  - *m*-1 ***knot points***, $t_4 \ldots t_{m+1}$
  - ***segments*** Q_i of the B-spline curve are
    - defined over a knot interval $[t_i, t_{i+1}]$
    - defined by 4 of the control points, $P_{i-3} \ldots P_i$
  - segments Q_i of the B-spline curve are blended together into smooth transitions via
    (the new & improved) ***blending functions***

Example: Creating a B-spline Curve Segment



B-splines: Knot Selection

- Instead of working with the parameter space $0 \le t \le 1$ use $t_{\min} \le t_0 \le t_1 \le t_2 \ldots \le t_{m-1} \le t_{\max}$
- The ***knot points***
  - joint points between curve segments, Q_i
  - Each has a *knot value*
  - *m-1* knots for *m+1* points



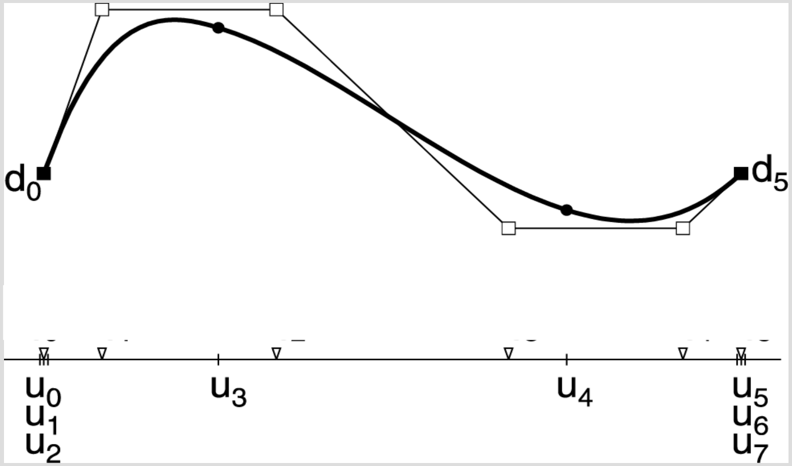
1994 Foley/VanDam/Finer/Huges/Phillips ICG

B-spline: Knot Sequences

- Even distribution of knots
  - *uniform* B-splines
  - Curve does not interpolate end points
    - first blending function not equal to 1 at t=0
- Uneven distribution of knots
  - *non-uniform* B-splines
  - Allows us to tie down the endpoints by repeating knot values (in Cox-deBoor, 0/0=1)
  - If a knot value is repeated, it increases the effect (weight) of the blending function at that point
  - If knot is repeated *d* times, blending function converges to 1 and the curve interpolates the control point

Creating a Non-Uniform B-spline: Knot Selection

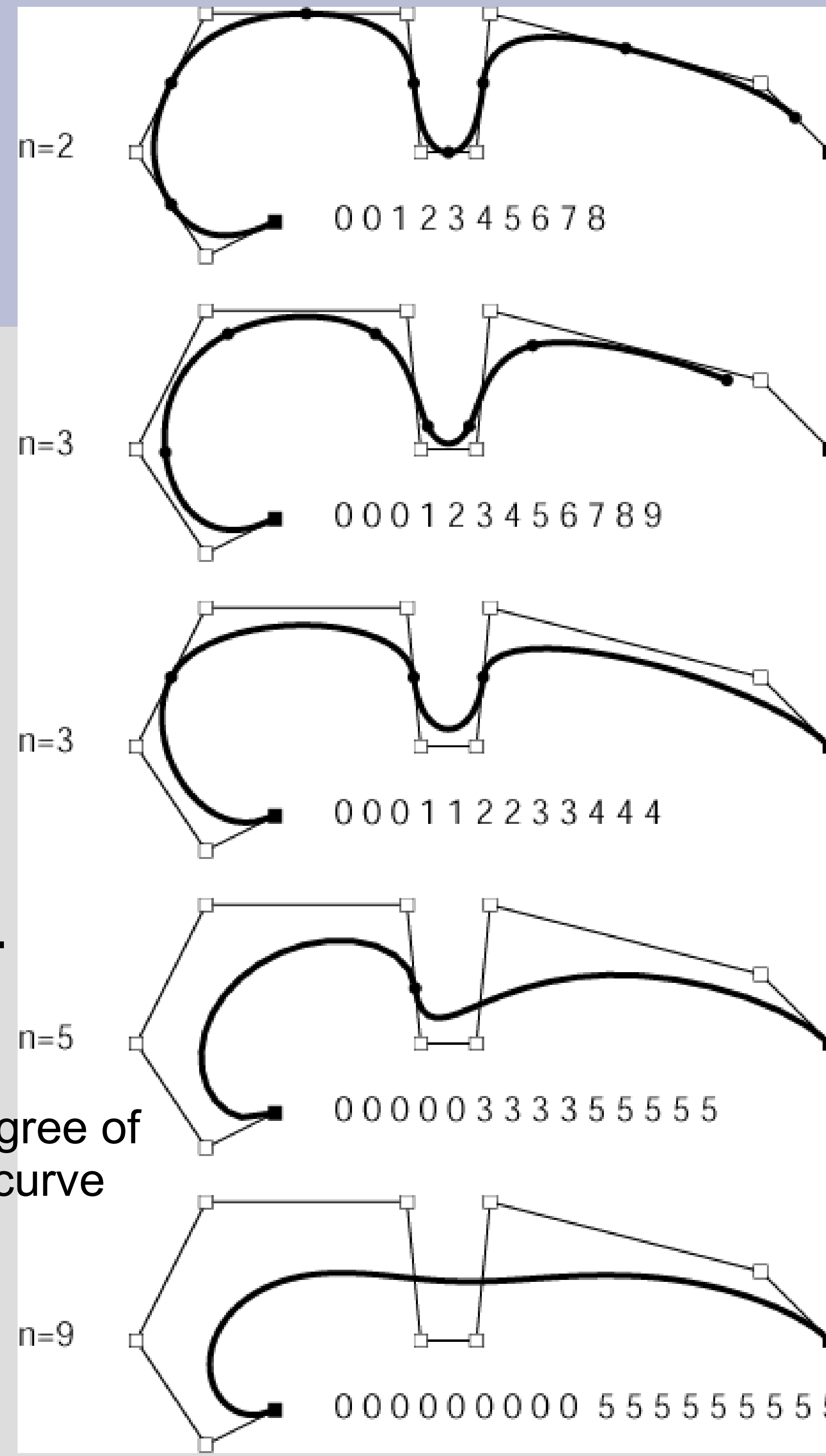
- Given curve of degree $d=3$, with $m+1$ control points $\mathbf{p}_0, \ldots, \mathbf{p}_m$
  - first, create $m-1+2(d-1)$ knot points
  - use knot values $(0,0,0,1,2,\ldots, m-2, m-1,m-1,m-1)$ (adding two extra 0's and $m-1$'s)
  - Note
    - Causes Cox-deBoor to give added weight in blending to the first and last points when t is near t_{min} and t_{max}



Pics/Math courtesy of G. Farin @ ASU

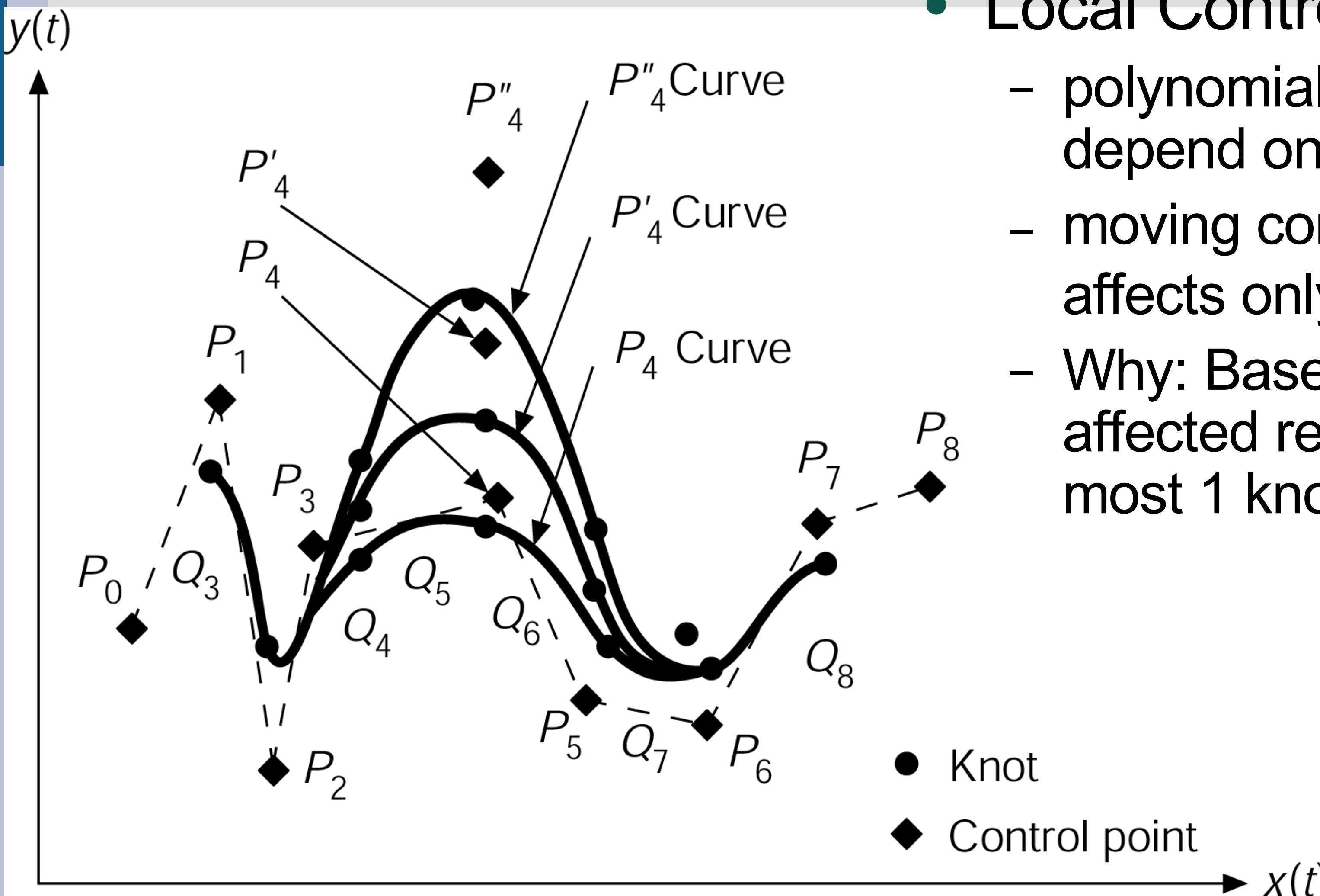
Watching Effects of Knot Selection

- 8 knot points (initially)
  - Note: knots are distributed parametrically based on *t*, hence why they “move”
- 10 control points
- Curves have as many segments as they have non-zero intervals in *u*



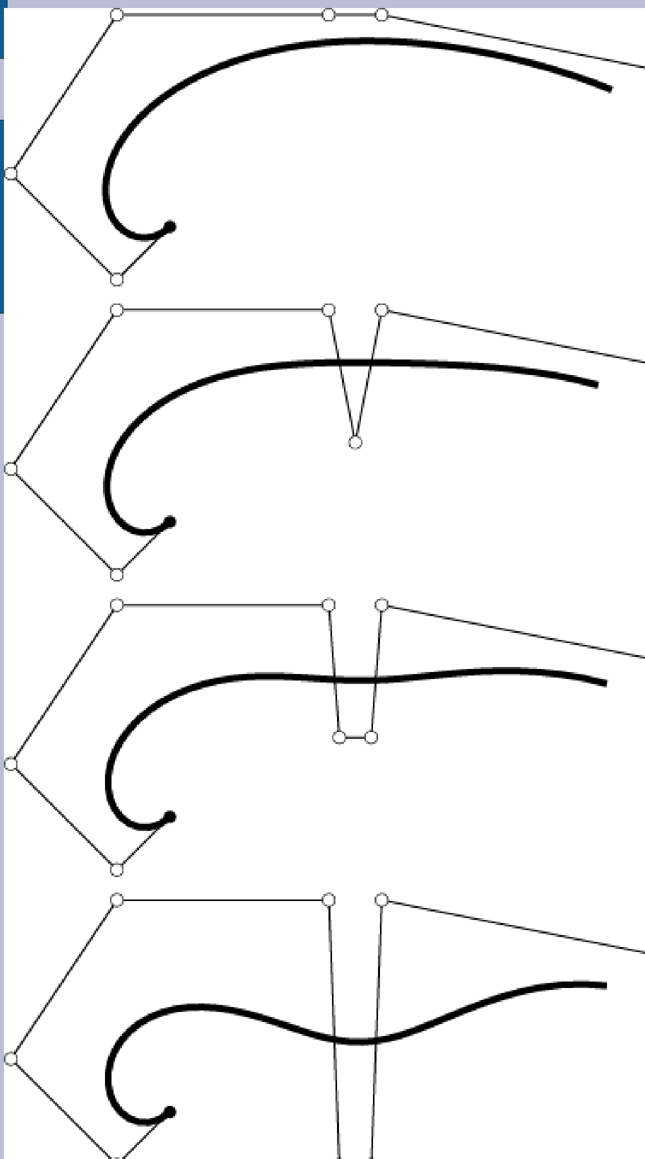
degree of curve

B-splines: Local Control Property

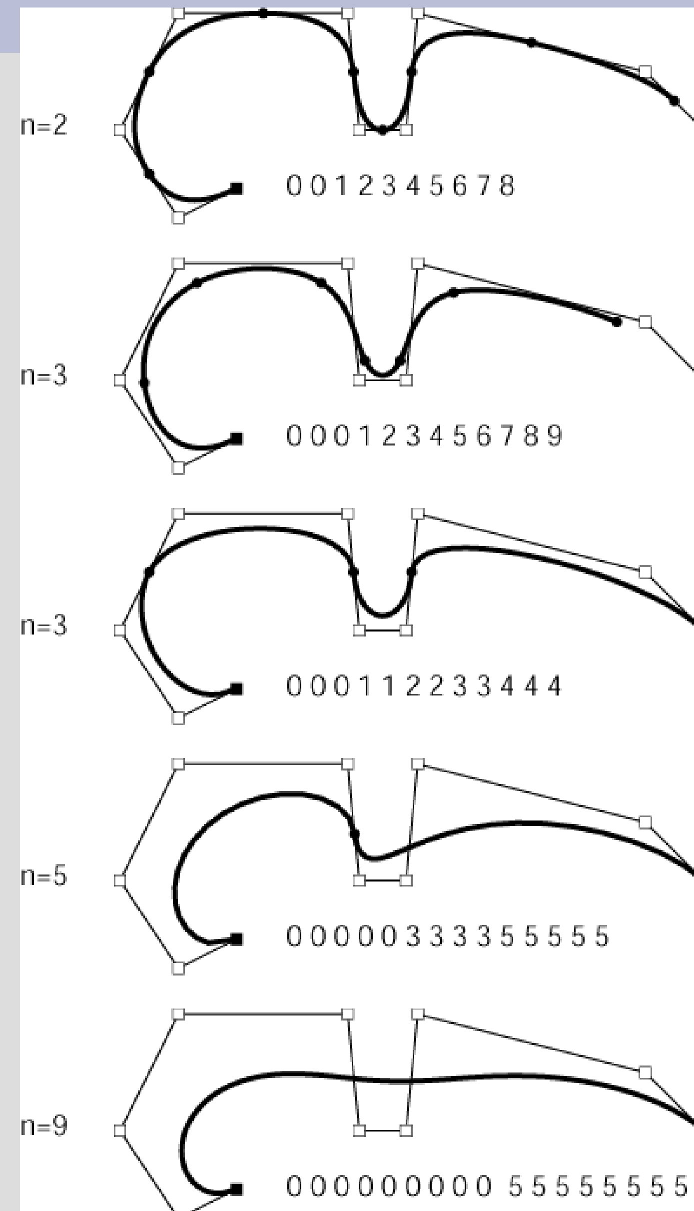


- Local Control
  - polynomial coefficients depend on a few points
  - moving control point (P_4) affects only local curve
  - Why: Based on curve def'n, affected region extends at most 1 knot point away

Control: Bézier vs B-splines



Observe the effect on the whole curve when controls are moved



B-splines: Setting the Options

- How to space the *knot points*?
  - **Uniform**
    - equal spacing of knots along the curve
  - **Non-Uniform**
- Which type of *parametric function*?
  - **Rational**
    - ***x(t), y(t), z(t)*** defined as ratio of cubic polynomials
  - **Non-Rational**