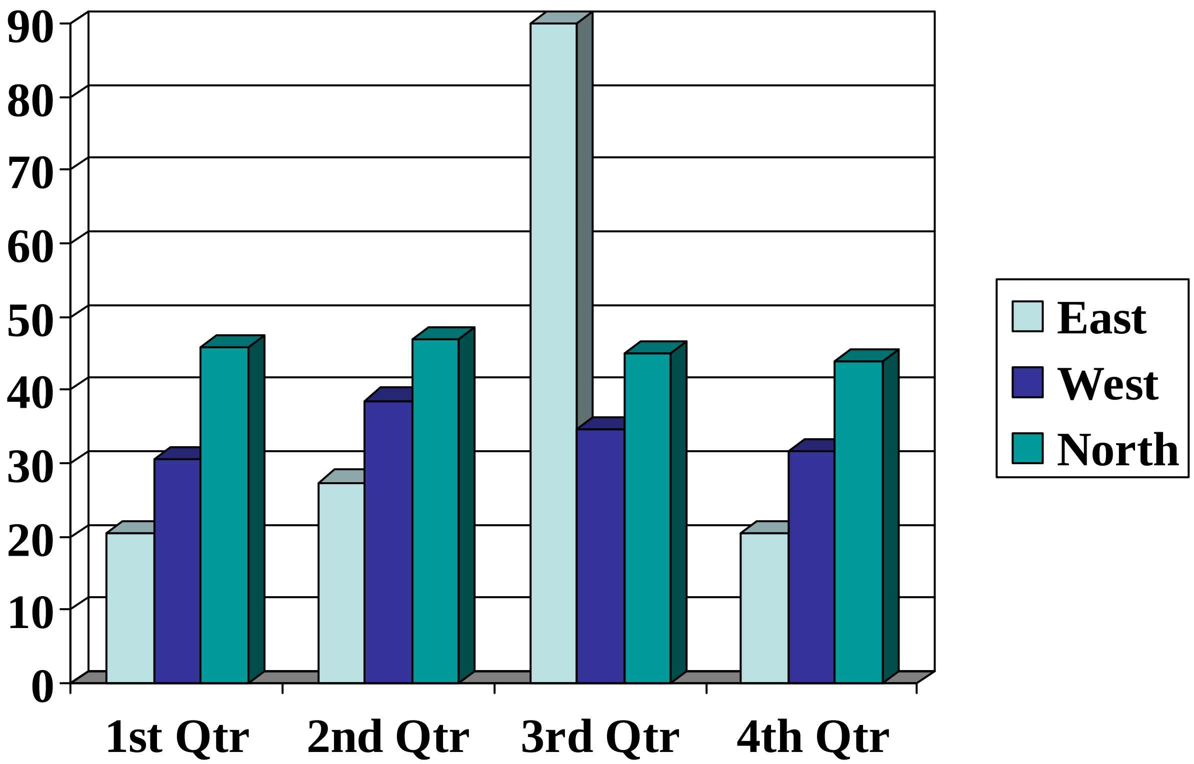
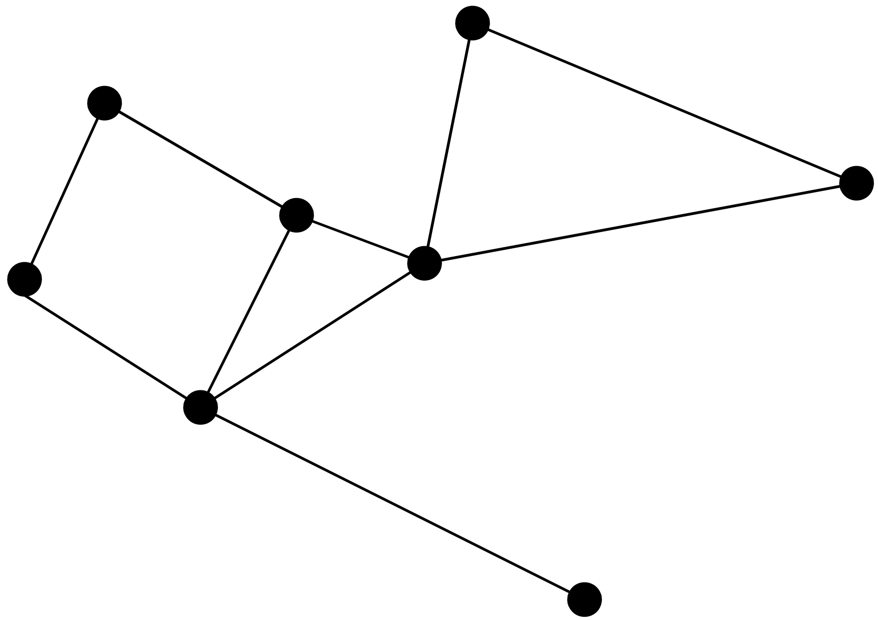
Graphs

CSIS 2226



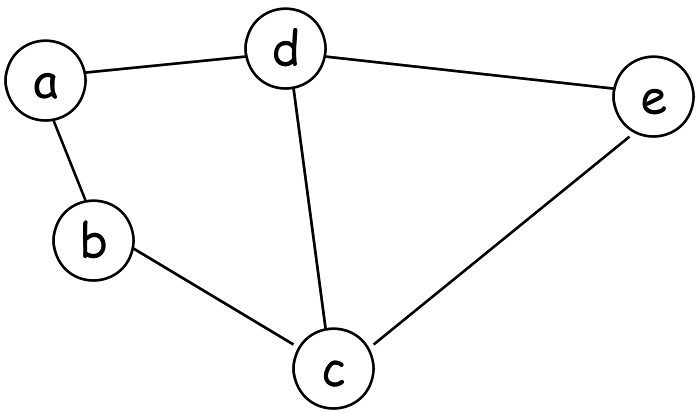
NOT ONE OF THESE!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!



One of these!

A Simple Graph

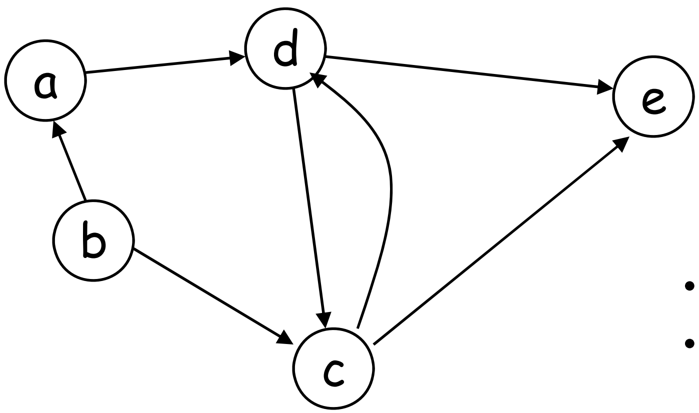
- G = (V,E)
  - V is set of vertices
  - E is set of edges



- V = {a,b,c,d,e}
- E = {(a,b),(a,d),(b,c),(c,d),(c,e),(d,e)}

A Directed Graph

- G = (V,E)
  - V is set of vertices
  - E is set of directed edges
    - directed pairs



- V = {a,b,c,d,e}
- E = {(a,d),(b,a),(b,c),(c,d),(c,e),(d,c),(d,e)}

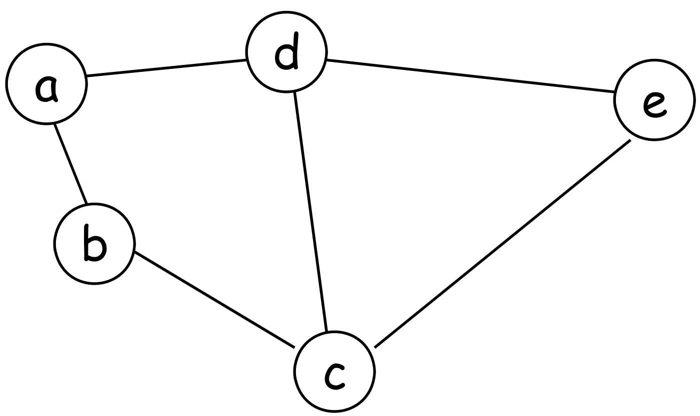
Applications

- computer networks
- telecomm networks
- scheduling (precedence graphs)
- transportation problems
- relationships
- chemical structures
- chemical reactions
- pert networks
- services (sewage, cable, …)
- WWW
- …

Terminology

- Vertex x is ***adjacent*** to vertex y if (x,y) is in E
  - c is adjacent to b, d, and e
- The ***degree*** of a vertex x is the number of edges incident on x
  - deg(d) = 3
  - note: degree aka ***valency***

- The graph has a ***degree sequence***
  - in this case 3,3,2,2,2



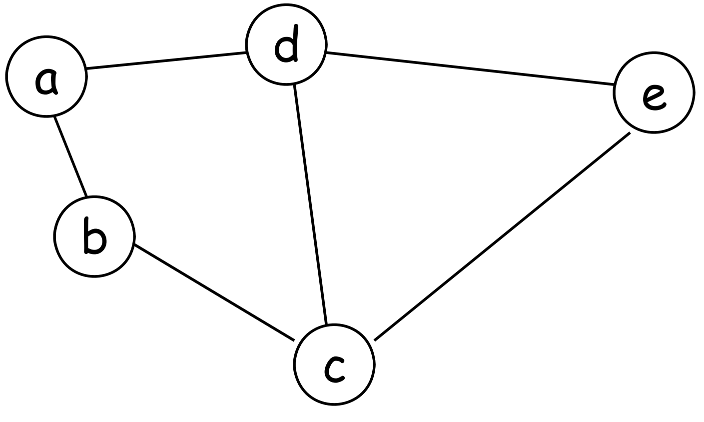
Handshaking Theorem (simple graph)

G = (V,E)

$$2e = \sum_{v \in V} \deg(v)$$

For an undirected graph G with e edges,
the sum of the degrees is 2e

Why?
- An edge (u,v) adds 1 to the degree of vertex u and vertex v
- Therefore edge(u,v) adds 2 to the sum of the degrees of G
- Consequently the sum of the degrees of the vertices is 2e



- 2e = deg(a) + deg(b) + deg(c) + deg(d) + deg(e)
- = 2 + 2 + 3 + 3 + 2
- = 12

Challenge: Draw a graph with degree sequence 2,2,2,1

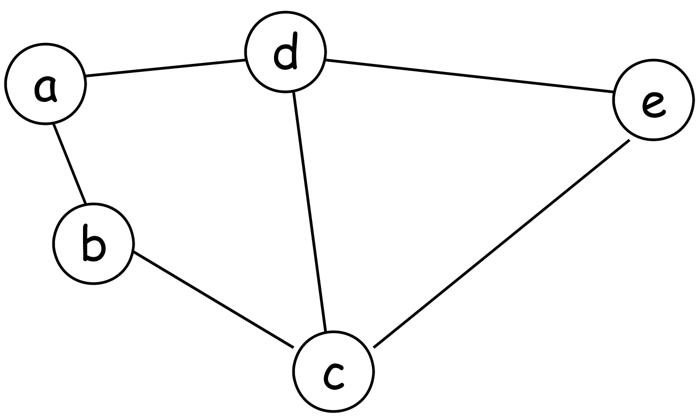
Handshaking Theorem (a consequence, for simple graphs)

There is an even number of vertices of odd degree

$$2e = \sum_{v \in V} \deg(v)$$

$$2e = \sum_{u \in OddDegVertices} \deg(u) + \sum_{v \in EvenDegVertices} \deg(w)$$

$$2k = \sum_{u \in OddDegVertices} \deg(u)$$



deg(d) = 3 and deg(c) = 3

Is there an algorithm for drawing a graph with a given degree sequence?

Yes, the Havel-Hakimi algorithm

The Havel-Hakimi Algorithm

Take as input a degree sequence S and determine if that sequence is graphical

That is, can we produce a graph with that degree sequence?

Assume the degree sequence is S $S = d_1, d_2, d_{3,} \cdots, d_n$

$d_i \geq d_{i+1}$

1. If any $d_i \geq n$ then fail
2. If there is an odd number of odd degrees then fail
3. If there is a $d_i < 0$ then fail
4. If all $d_i = 0$ then report success
5. Reorder S into non-increasing order
6. Let $k = d_1$
7. Remove d_1 from S.
8. Subtract 1 from the first k terms remaining of the new sequence
9. Go to step 3 above

Note: steps 1 and 2 are a pre-process

3. If there is a $d_i < 0$ then fail

4. If all $d_i = 0$ then report success

5. Reorder S into non-increasing order

6. Let $k = d_1$

7. Remove d_1 from S.

8. Subtract 1 from the first k terms remaining of the new sequence

9. Go to step 3 above

S = 4,3,3,3,1

S = 4,3,3,3,1

3. If there is a $d_i < 0$ then fail

4. If all $d_i = 0$ then report success

5. Reorder S into non-increasing order

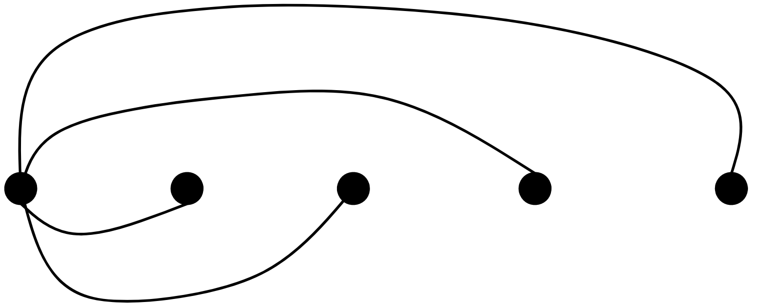
6. Let $k = d_1$

7. Remove d_1 from S.

8. Subtract 1 from the first k terms remaining of the new sequence

9. Go to step 3 above

S = 4,3,3,3,1



S = 2,2,2,0

3. If there is a $d_i < 0$ then fail

4. If all $d_i = 0$ then report success

5. Reorder S into non-increasing order

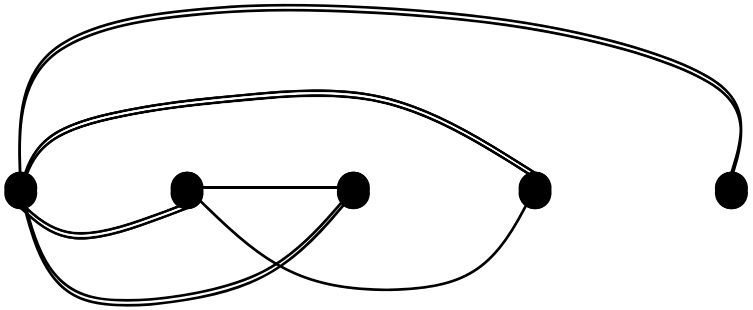
6. Let $k = d_1$

7. Remove d_1 from S.

8. Subtract 1 from the first k terms remaining of the new sequence

9. Go to step 3 above

S = 4,3,3,3,1



S = 1,1,0

3. If there is a $d_i < 0$ then fail

4. If all $d_i = 0$ then report success

5. Reorder S into non-increasing order

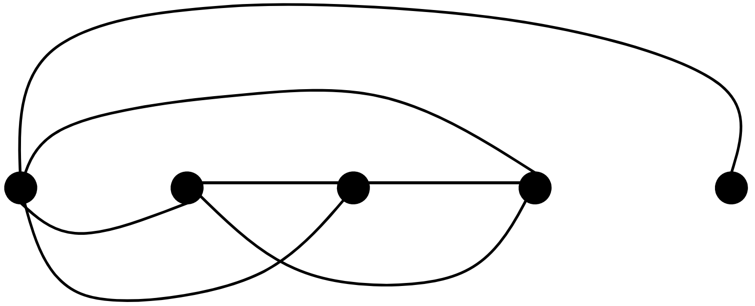
6. Let $k = d_1$

7. Remove d_1 from S.

8. Subtract 1 from the first k terms remaining of the new sequence

9. Go to step 3 above

S = 4,3,3,3,1



S = 0,0

3. If there is a $d_i < 0$ then fail

4. If all $d_i = 0$ then report success

5. Reorder S into non-increasing order

6. Let $k = d_1$

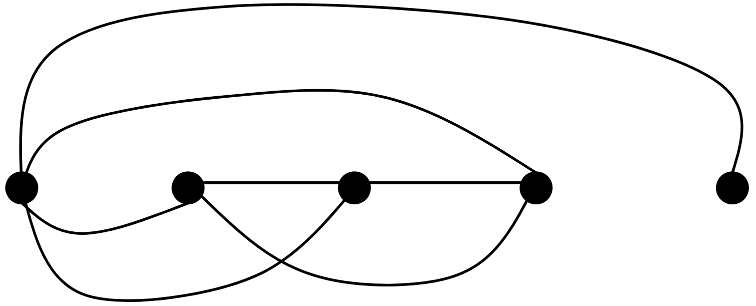
7. Remove d_1 from S.

8. Subtract 1 from the first k terms remaining of the new sequence

9. Go to step 3 above

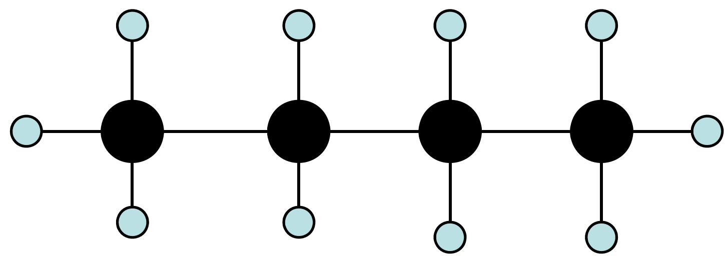
S = 4,3,3,3,1

Report Success



4,4,4,4,1,1,1,1,1,1,1,1,1,1

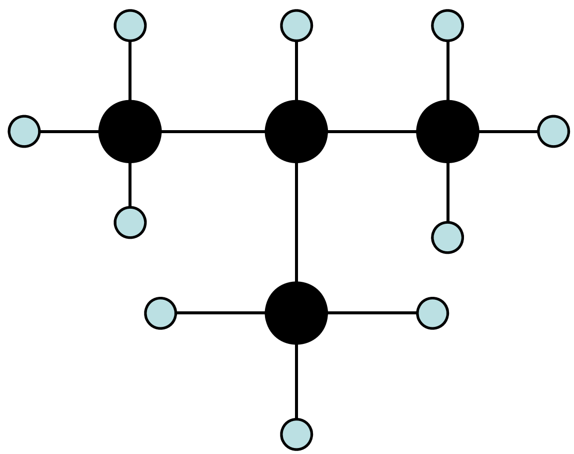
C_4H_{10}



Alternatively

4,4,4,4,1,1,1,1,1,1,1,1,1,1

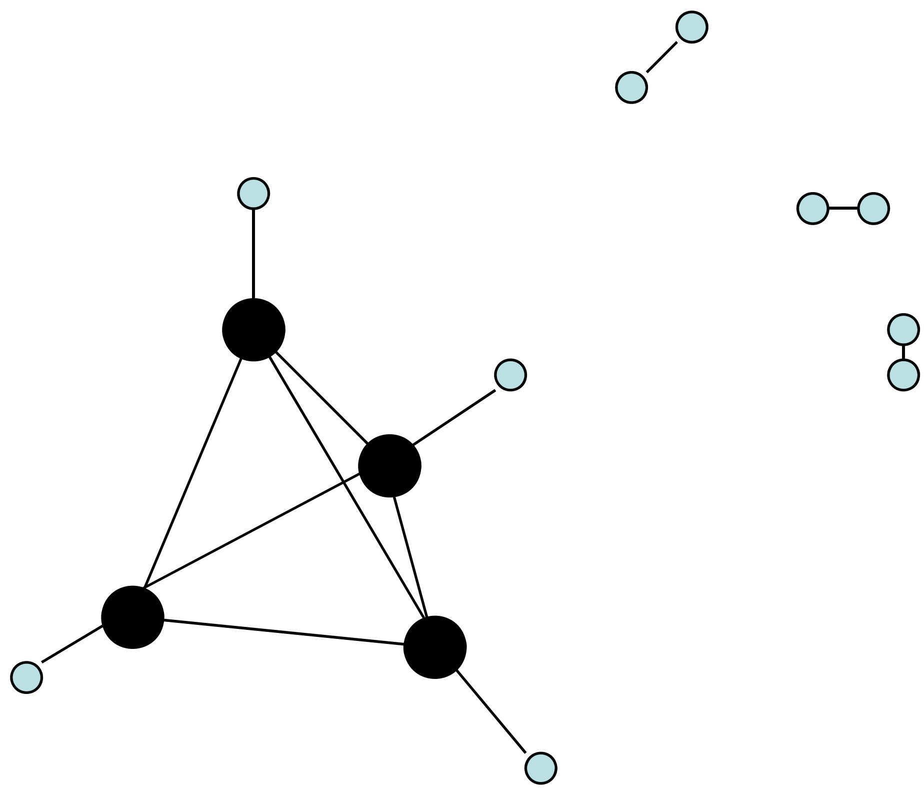
$$C_4H_{10}$$



Havel-Hakimi produces the following

4,4,4,4,1,1,1,1,1,1,1,1,1,1

$$C_4H_4 + 3H_2$$



The hypothetical hydrocarbon **Vinylacetylene**

So?

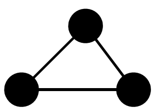
Well, we have demonstrated that the HH algorithm doesn't always produce
A connected graph.

We have also shown that by representing molecules as simple graphs and using an algorithm to model this graph we might get some unexpected results, maybe something new!

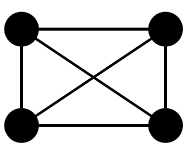
(Some) Special Graphs

K_1

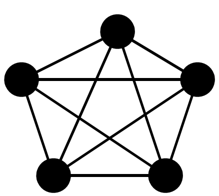
K_2



K_3



K_4



K_5

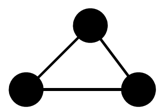
$$G = (V, E)$$

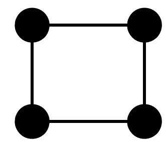
$$n = |V|$$

$$|E| = \frac{n(n-1)}{2}$$

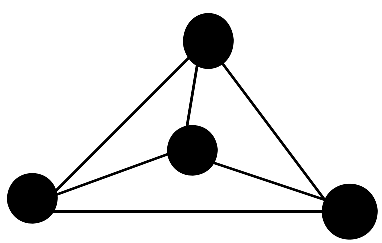
Cliques

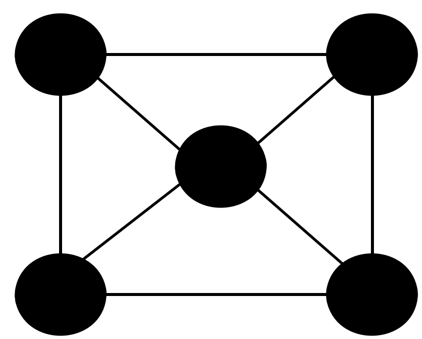
Cycles





Wheels



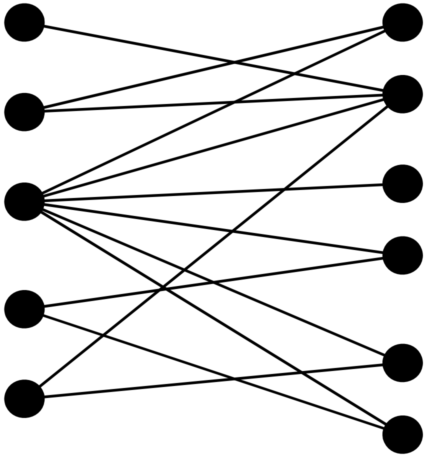


Bipartite Graphs

Vertex set can be divided into 2 disjoint sets

$$V = V_1 \cup V_2$$

$$(v,w) \in E \rightarrow (v \in V_1 \wedge w \in V_2) \oplus (v \in V_2 \wedge w \in V_1)$$



Other Kinds of Graphs

- multigraphs
  - may have multiple edges between a pair of vertices
  - in telecomms, these might be redundant links, or extra capacity

- pseudographs
  - a multigraphs, but edges (v,v) are allowed

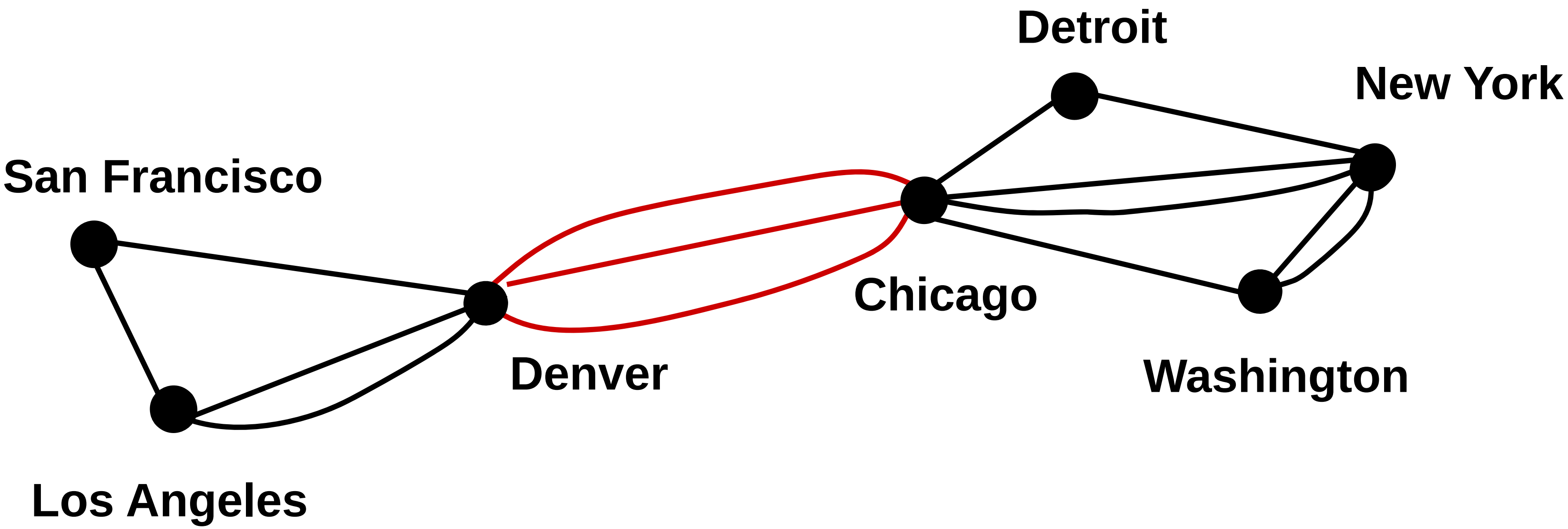
- hypergraph
  - hyperedges, involving more than a pair of vertices

A Non-Simple Graph

Definition 2. **In a multigraph *G = (V, E)* two or more edges may connect the same pair of vertices.**

A Multigraph

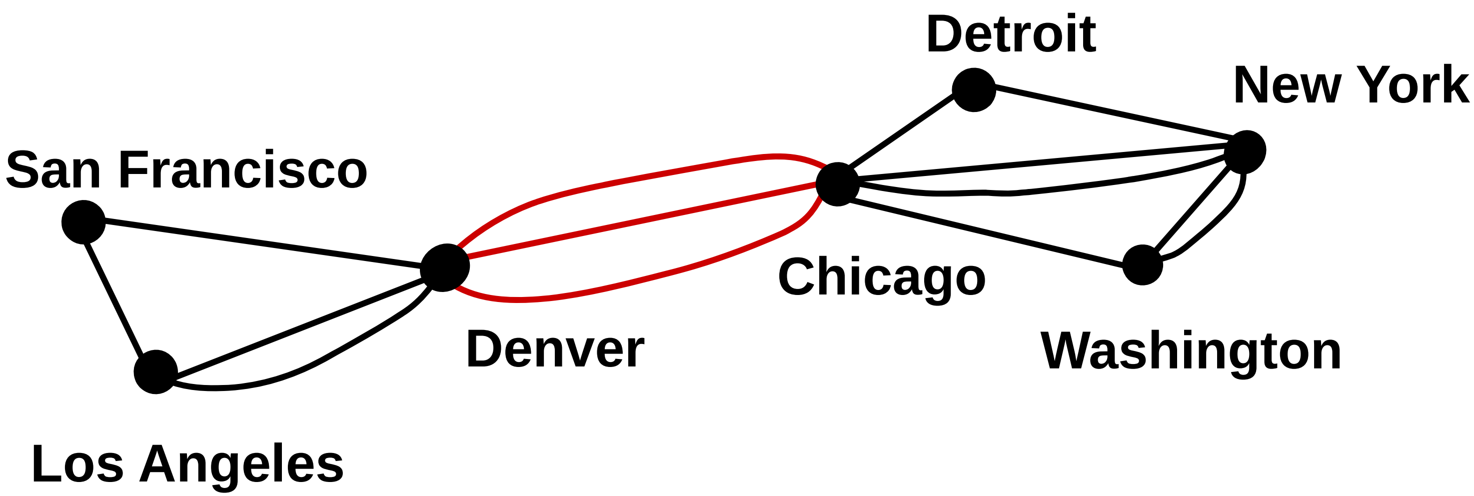
THERE CAN BE MULTIPLE TELEPHONE LINES BETWEEN TWO COMPUTERS IN THE NETWORK.



Another Non-Simple Graph

Definition 3. **In a pseudograph *G = (V, E)* two or more edges may connect the same pair of vertices, and in addition, an edge may connect a vertex to itself.**

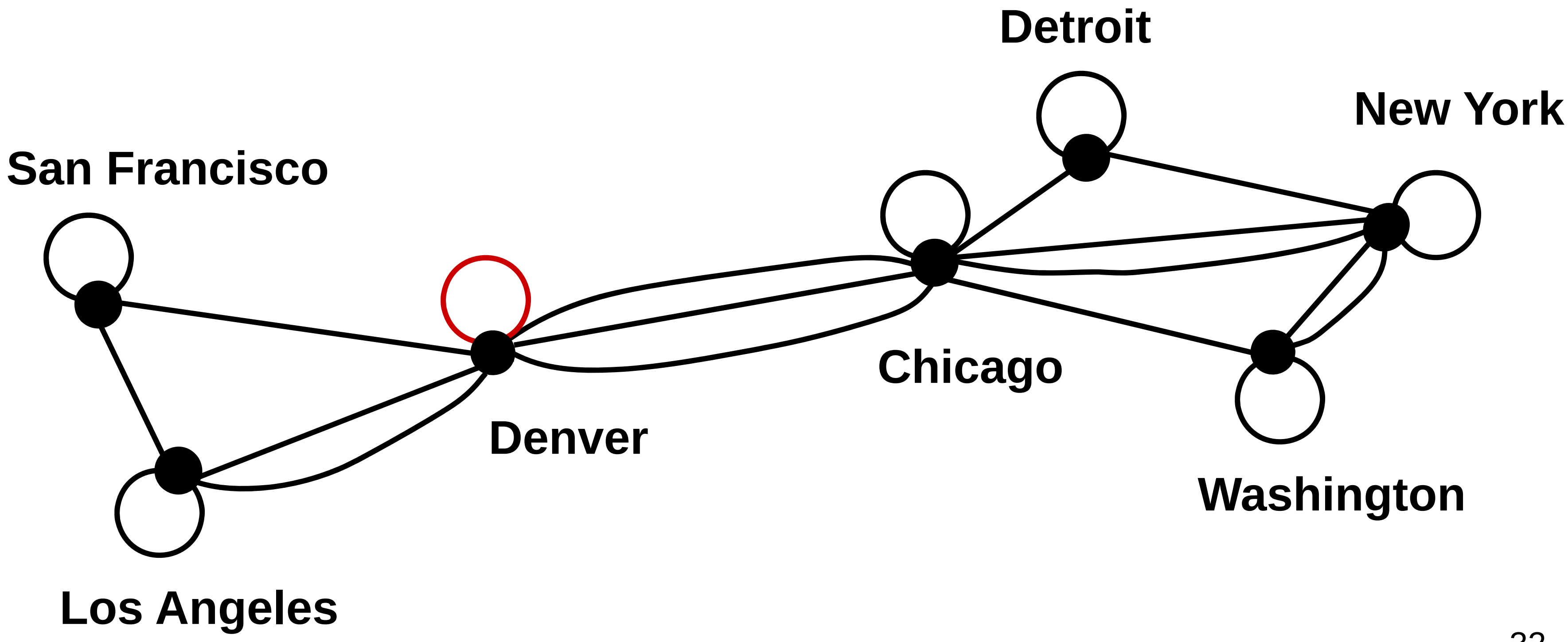
Multiple Edges



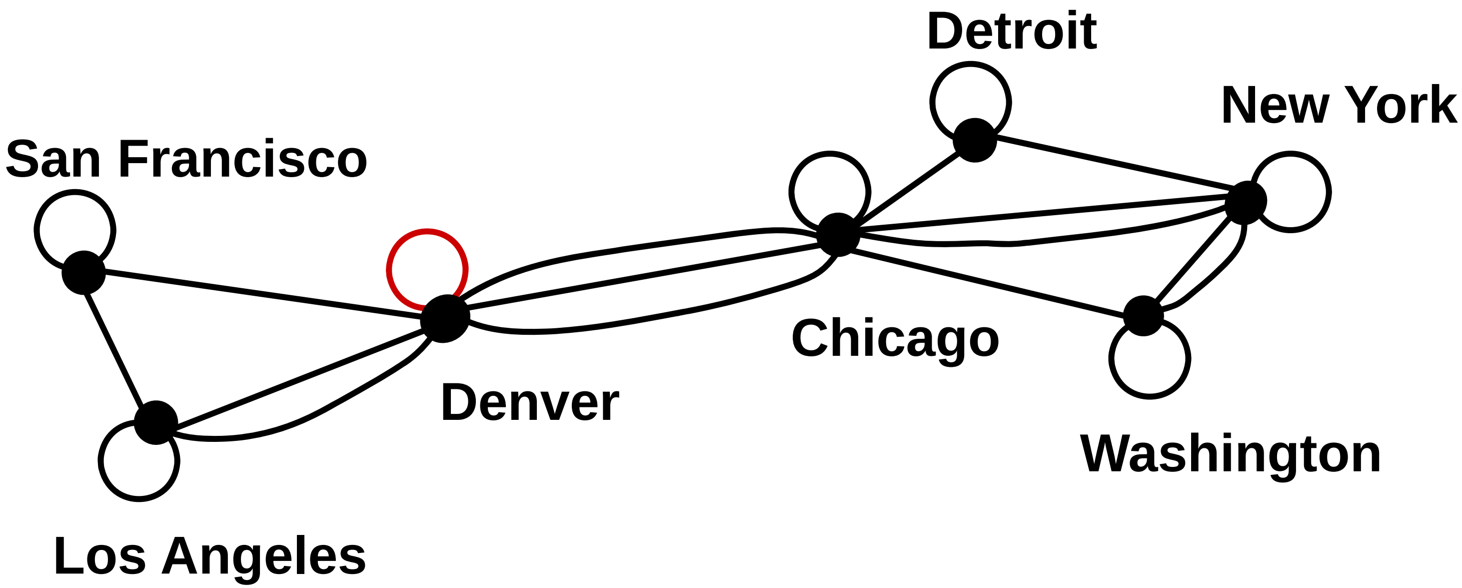
Two edges are called *multiple or parallel edges* if they connect the same two distinct vertices.

A Pseudograph

THERE CAN BE TELEPHONE LINES IN THE NETWORK FROM A COMPUTER TO ITSELF (for diagnostic use).

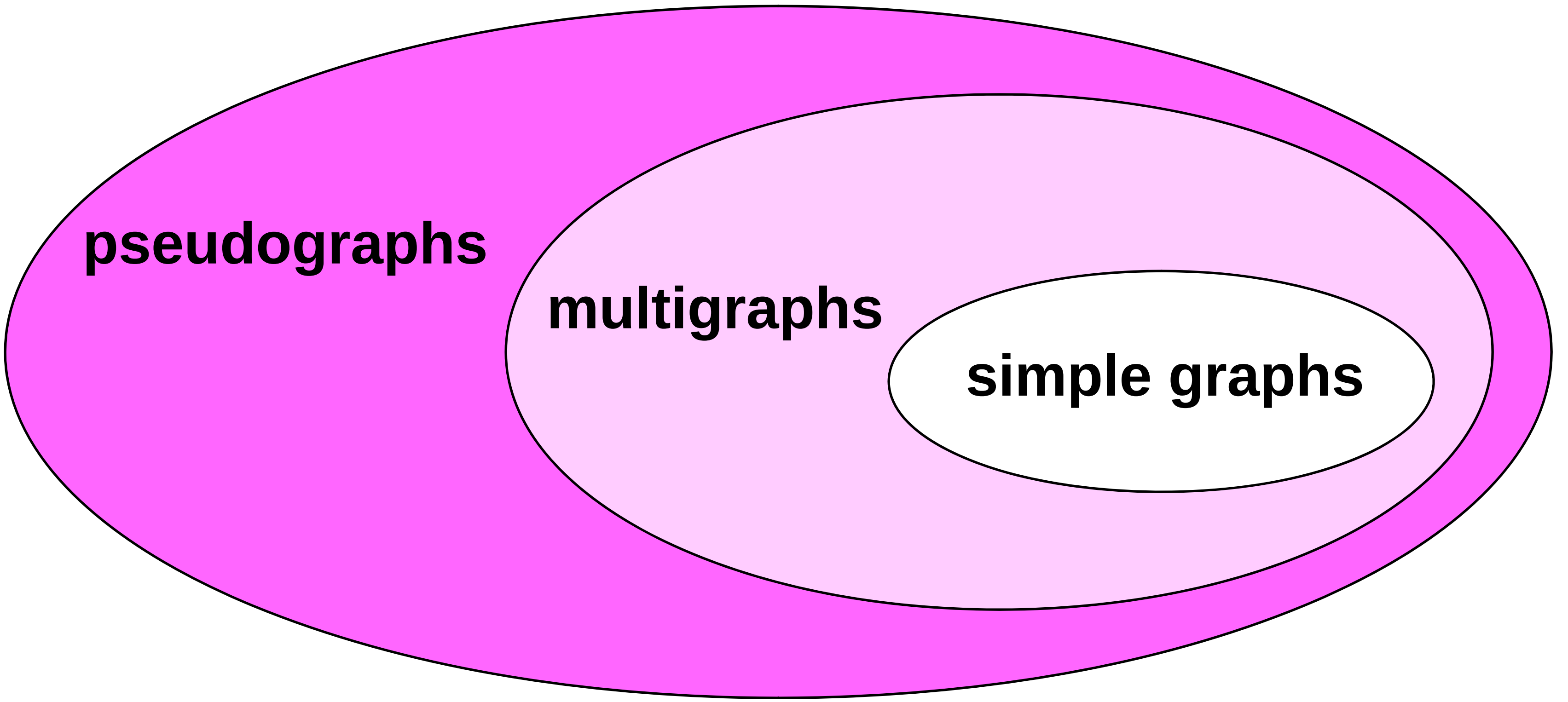


Loops

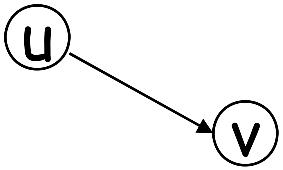


An edge is called a *loop*
if it connects a vertex to itself.

Undirected Graphs



Directed Graphs



- (u,v) is a directed edge
- u is the initial vertex
- v is the terminal or end vertex

- the in-degree of a vertex
  - number of edges with v as terminal vertex

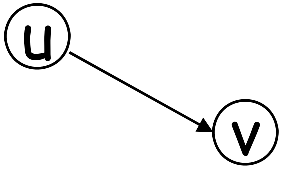
$$\deg^{+}(v)$$

- the out-degree of a vertex
  - number of edges with v as initial vertex

$$\deg^{-}(v)$$

Directed Graphs

- (u,v) is a directed edge
- u is the initial vertex
- v is the terminal or end vertex



$$\sum_{v \in V} \deg^{+}(v) = \sum_{v \in V} \deg^{-}(v) = |E|$$

Each directed edge (v,w) adds 1 to the out-degree of one vertex and adds 1 to the in-degree of another

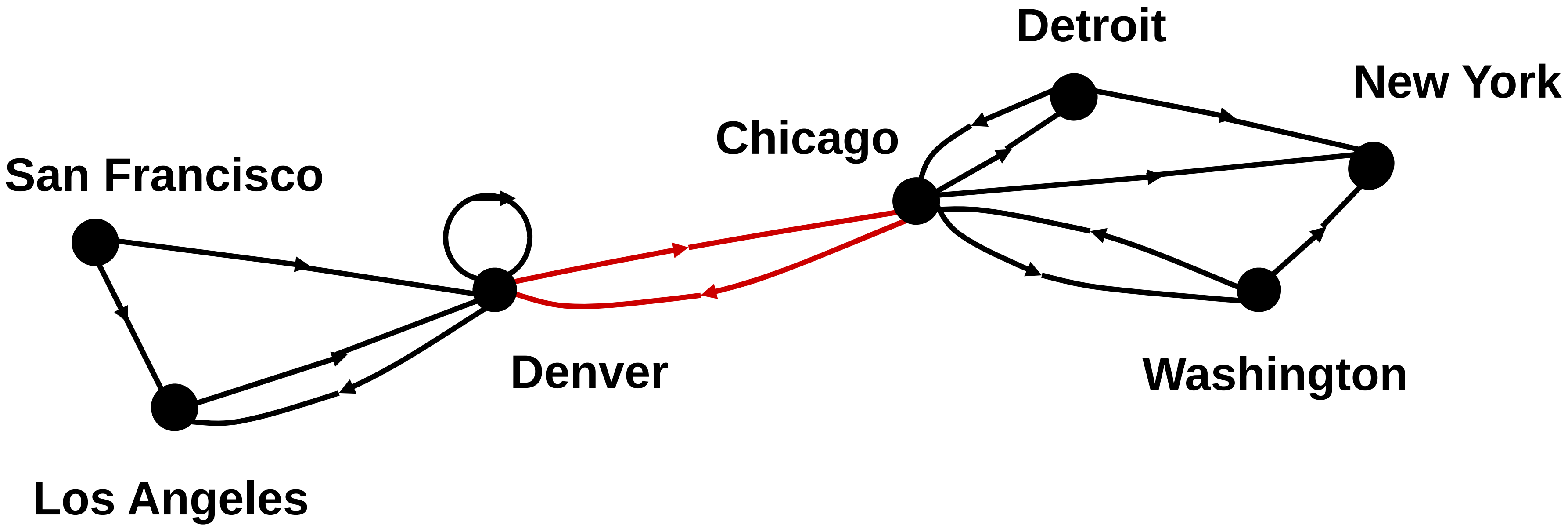
A Directed Graph

Definition 4. **In a directed graph** ***G = (V, E)*** **the edges are ordered pairs of (not necessarily distinct) vertices.**

A Directed Graph

SOME TELEPHONE LINES IN THE NETWORK MAY OPERATE IN ONLY ONE DIRECTION .

Those that operate in two directions are represented by pairs of edges in opposite directions.

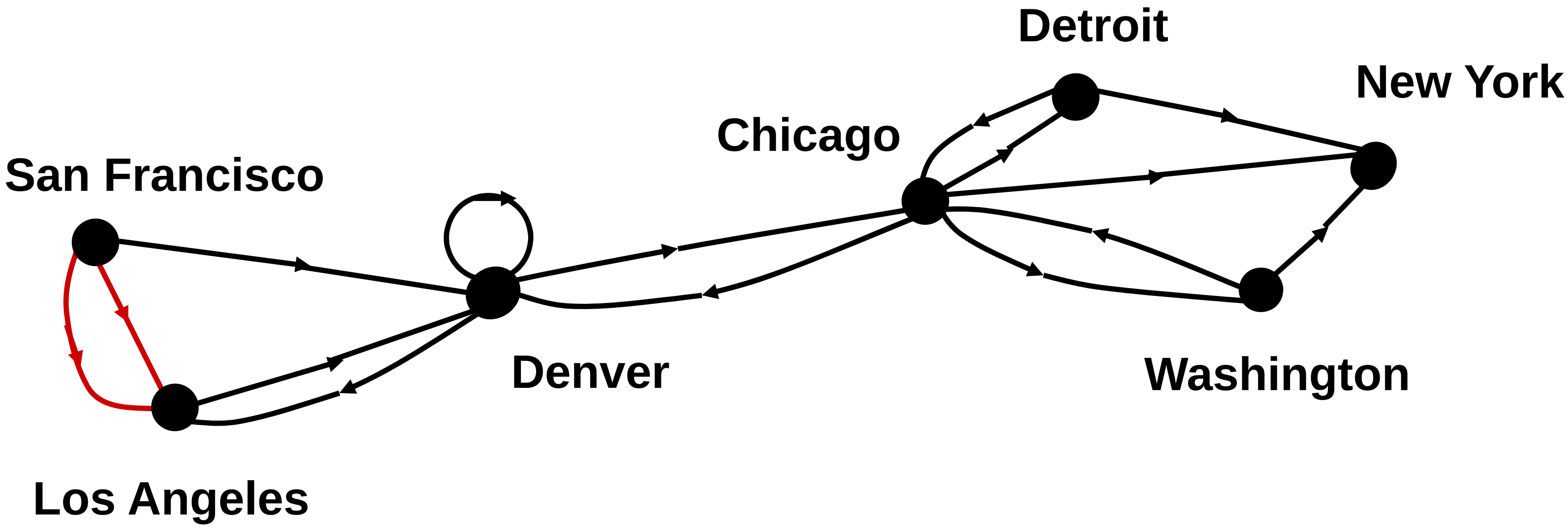


A Directed Multigraph

Definition 5. In a directed multigraph *G = (V, E)* the edges are ordered pairs of (not necessarily distinct) vertices, and in addition there may be multiple edges.

A Directed Multigraph

THERE MAY BE SEVERAL ONE-WAY LINES
IN THE SAME DIRECTION FROM ONE COMPUTER
TO ANOTHER IN THE NETWORK.

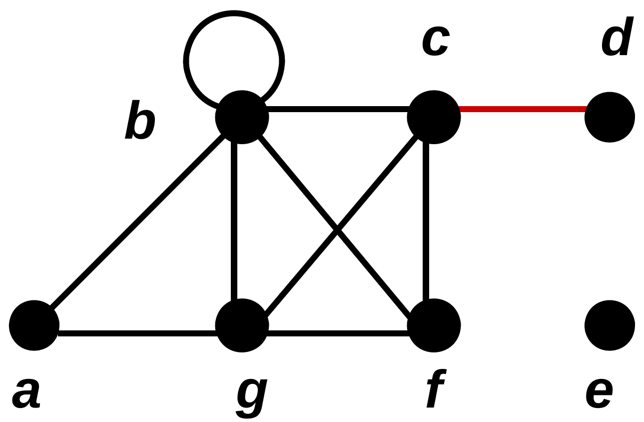


Types of Graphs

TYPE	EDGES	MULTIPLE EDGES ALLOWED?	LOOPS ALLOWED?
Simple graph	Undirected	NO	NO
Multigraph	Undirected	YES	NO
Pseudograph	Undirected	YES	YES
Directed graph	Directed	NO	YES
Directed multigraph	Directed	YES	YES

Degree of a vertex

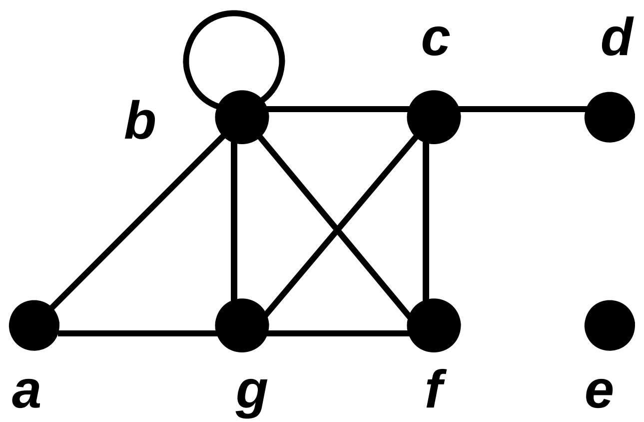
Definition 1. **The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.**



deg(*d*) = 1

Degree of a vertex

Definition 1. **The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.**

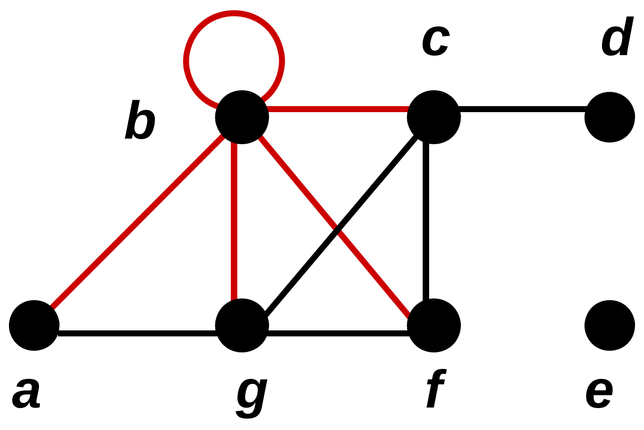


deg(*e*) = 0

Degree of a vertex

Definition 1. **The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.**

deg(*b*) = 6

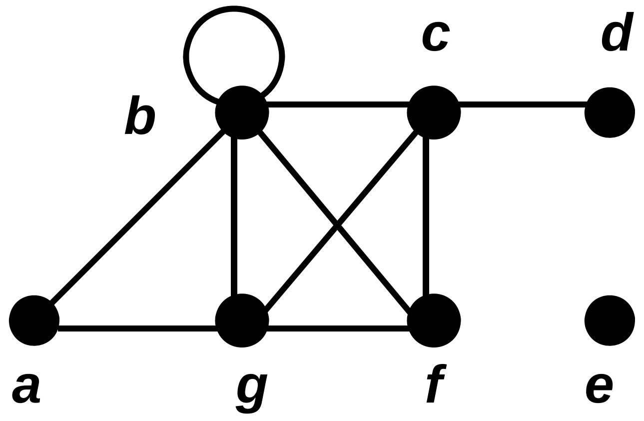


Degree of a vertex

Find the degree of all the other vertices.

deg(*a*)   deg(c)   deg(f)   deg(*g*)

deg(*b*) = 6



deg(*d*) = 1

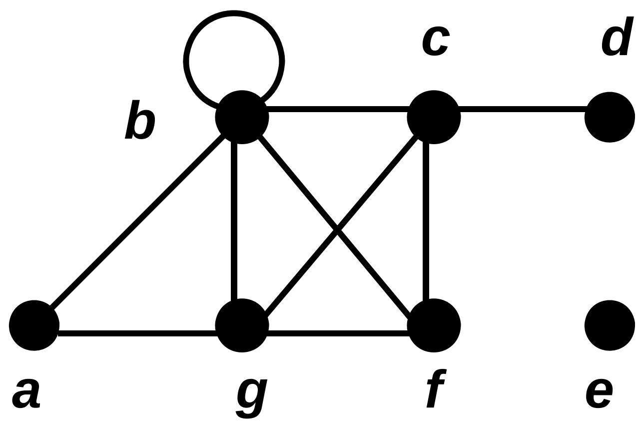
deg(*e*) = 0

Degree of a vertex

Find the degree of all the other vertices.

deg(*a*) = 2   deg(c) = 4   deg(f) = 3   deg(*g*) = 4

deg(*b*) = 6



deg(*d*) = 1

deg(*e*) = 0

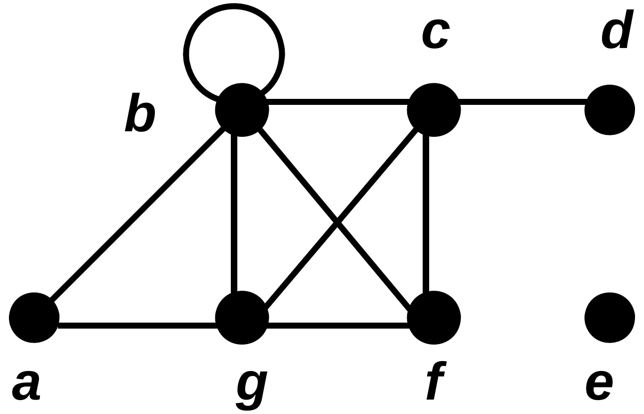
Degree of a vertex

Find the degree of all the other vertices.

deg(*a*) = 2   deg(c) = 4   deg(f) = 3   deg(*g*) = 4

TOTAL of degrees = 2 + 4 + 3 + 4 + 6 + 1 + 0 = 20

deg(*b*) = 6



deg(*d*) = 1

deg(*e*) = 0

Degree of a vertex

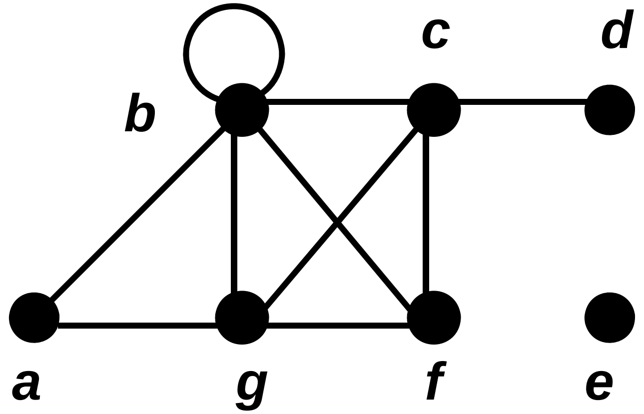
Find the degree of all the other vertices.

deg(*a*) = 2   deg(c) = 4   deg(f) = 3   deg(*g*) = 4

TOTAL of degrees = 2 + 4 + 3 + 4 + 6 + 1 + 0 = 20

TOTAL NUMBER OF EDGES = 10

deg(*b*) = 6



deg(*d*) = 1

deg(*e*) = 0

New Graphs from Old?

We can have a subgraph

$$G = (V, E)$$
$$H = (W, F)$$
$$W \subseteq V$$
$$F \subseteq E$$

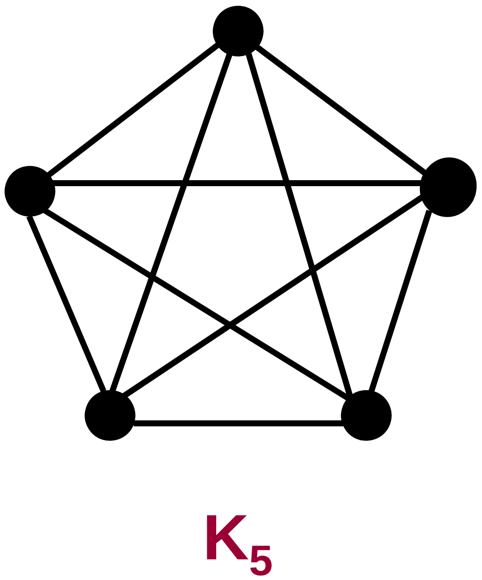
We can have a union of graphs

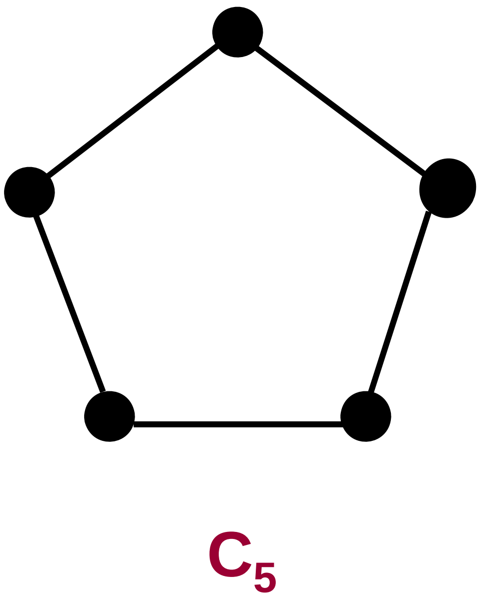
$$G_1 = (V_1, E_1)$$
$$G_2 = (V_2, E_2)$$
$$G_3 = G_1 \cup G_2$$
$$G_3 = (V_1 \cup V_2, E_1 \cup E_2)$$

Subgraph

Definition 6. **A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.**

C_5 is a subgraph of K_5

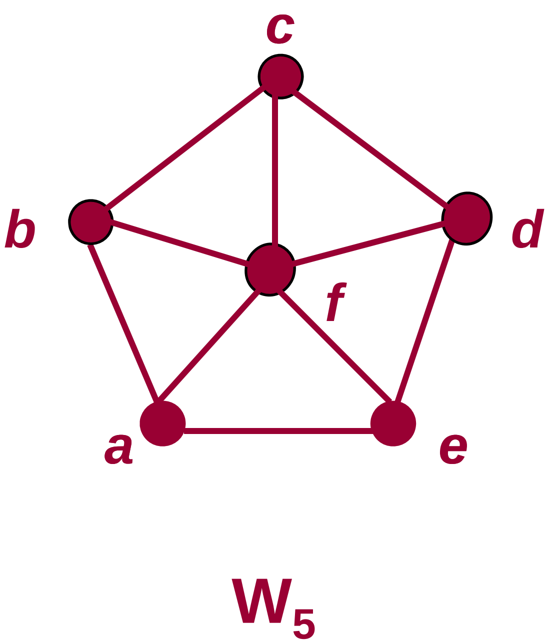


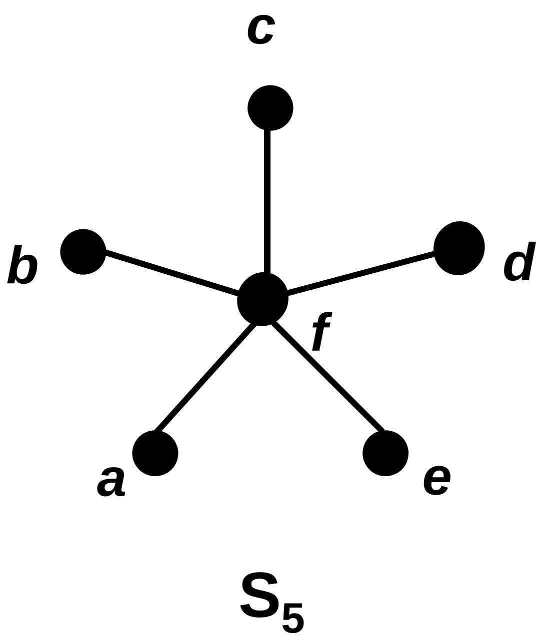


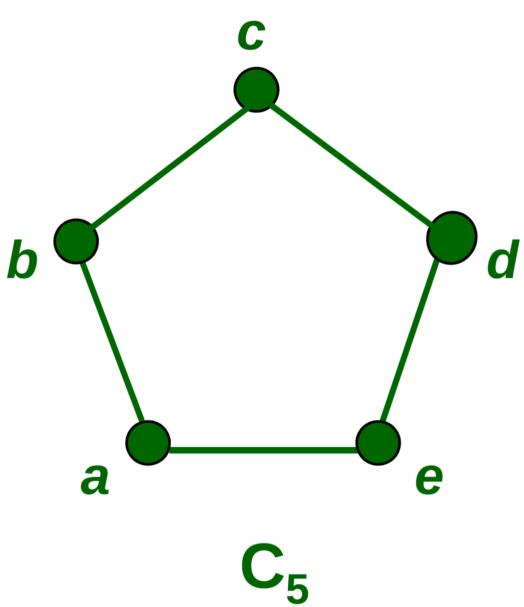
Union

Definition 7. **The union of 2 simple graphs $G_1 = (V_1 , E_1)$ and $G_2 = (V_2 , E_2)$ is the simple graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The union is denoted by $G_1 \cup G_2$.**

W_5 is the union of S_5 and C_5







Representing a Graph (Rosen 7.3, pages 456 to 463)

Adjacency Matrix: a 0/1 matrix A

$$(i, j) \in E \leftrightarrow a_{i,j} = 1$$

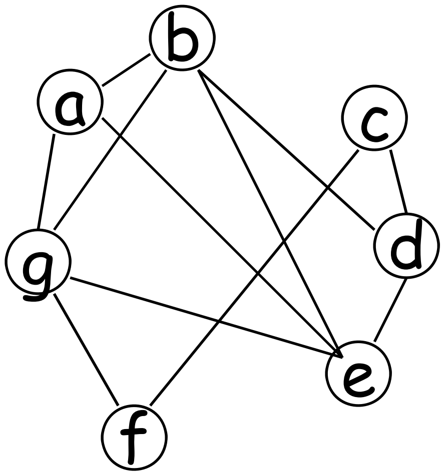
NOTE: A is symmetric for simple graphs!

$$(i, j) \in E \leftrightarrow a_{i,j} = 1 = a_{j,i}$$

NOTE: simple graphs do not have loops (v,v)

$$\forall i (a_{i,i} = 0)$$

Representing a Graph (Rosen 8.3)



$A =$

	a	b	c	d	e	f	g
a	0	1	0	0	1	0	1
b	1	0	0	1	1	0	1
c	0	0	0	1	0	1	0
d	0	1	1	0	1	0	0
e	1	1	0	1	0	0	1
f	0	0	1	0	0	0	1
g	1	1	0	0	1	1	0

$$A^2$$

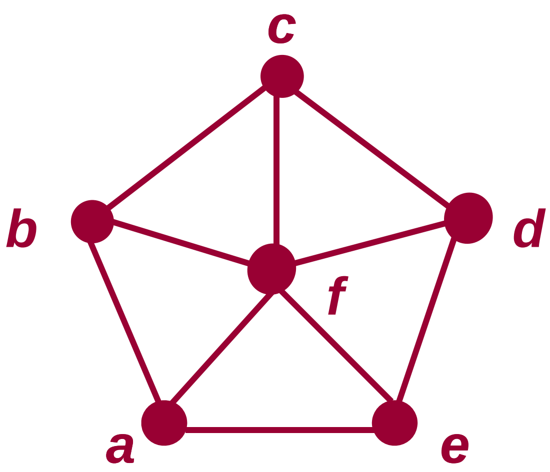
What's that then?

Adjacency Matrix

A simple graph $G = (V, E)$ with n vertices can be represented by its adjacency matrix, A, where entry a_{ij} in row i and column j is

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge in G,} \\ 0 & \text{otherwise.} \end{cases}$$

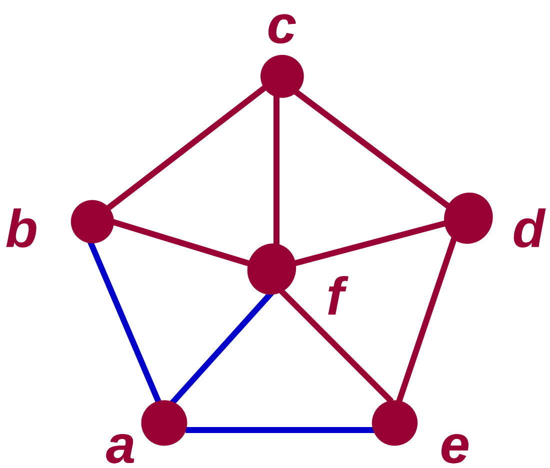
Finding the adjacency matrix



W_5

This graph has 6 vertices a, b, c, d, e, f. We can arrange them in that order.

Finding the adjacency matrix



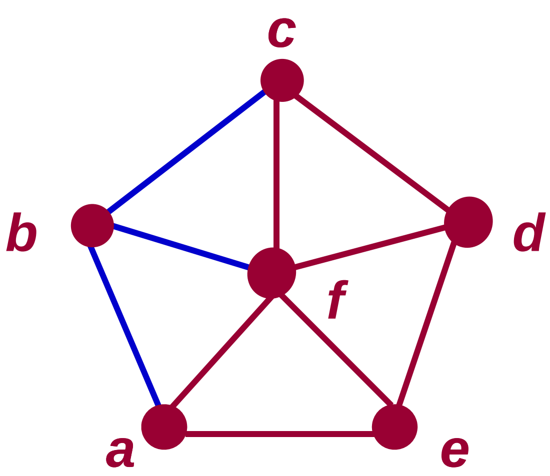
W$_5$

TO

FROM	a	b	c	d	e	f
a	**0**	**1**	**0**	**0**	**1**	**1**
b						
c						
d						
e						
f						

There are edges from a to b, from a to e, and from a to f

Finding the adjacency matrix



W_5

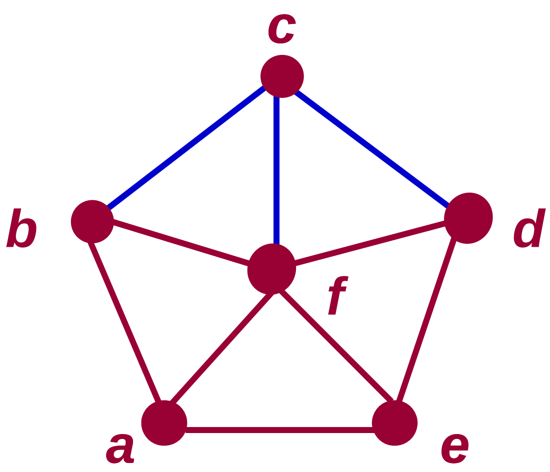
TO

FROM

	a	b	c	d	e	f
a	**0**	**1**	**0**	**0**	**1**	**1**
b	**1**	**0**	**1**	**0**	**0**	**1**
c						
d						
e						
f						

There are edges from b to a, from b to c, and from b to f

Finding the adjacency matrix



W_5

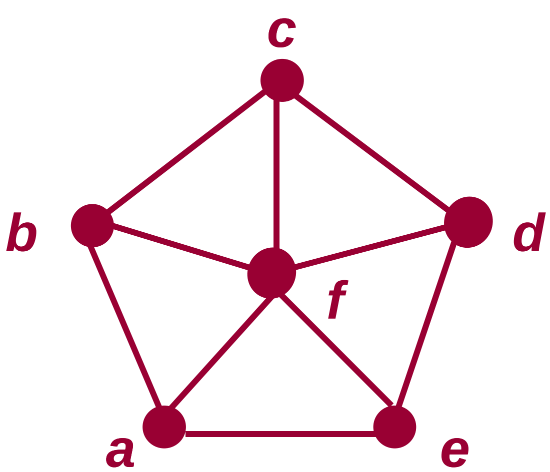
To

FROM

	a	b	c	d	e	f
a	0	1	0	0	1	1
b	1	0	1	0	0	1
c	0	1	0	1	0	1
d						
e						
f						

There are edges from c to b, from c to d, and from c to f

Finding the adjacency matrix



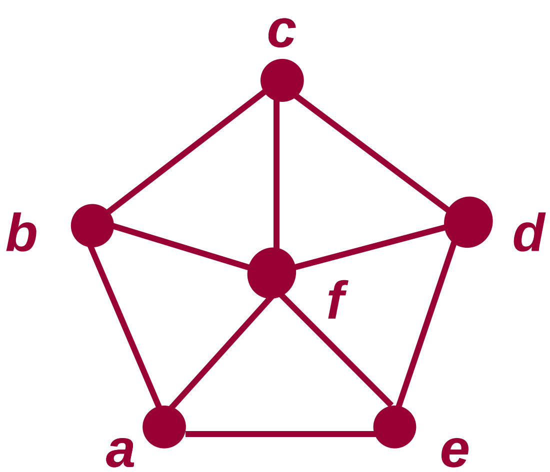
W_5

TO

FROM	a	b	c	d	e	f
a	**0**	**1**	**0**	**0**	**1**	**1**
b	**1**	**0**	**1**	**0**	**0**	**1**
c	**0**	**1**	**0**	**1**	**0**	**1**
d						
e						
f						

COMPLETE THE ADJACENCY MATRIX . . .

Finding the adjacency matrix



W_5

TO

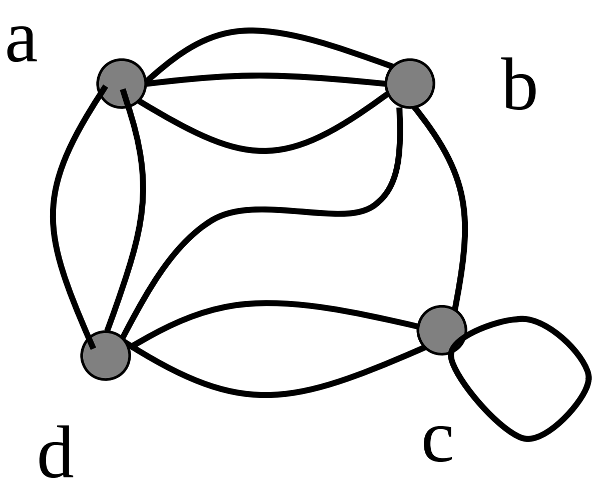
FROM

	a	b	c	d	e	f
a	0	1	0	0	1	1
b	1	0	1	0	0	1
c	0	1	0	1	0	1
d	0	0	1	0	1	1
e	1	0	0	1	0	1
f	1	1	1	1	1	0

Notice that this matrix is symmetric. That is $a_{ij} = a_{ji}$ Why?

Adjacency Matrices for pseudo graph

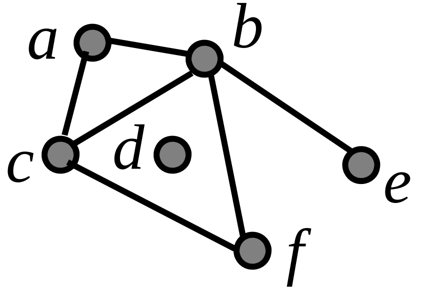
- Matrix **A**=[a_{ij}], where a_{ij} is the number of edges that are associated to $\{v_i, v_j\}$



	a	b	c	d
a	0	3	0	2
b	3	0	1	1
c	0	1	1	2
d	2	1	2	0

Adjacency Lists

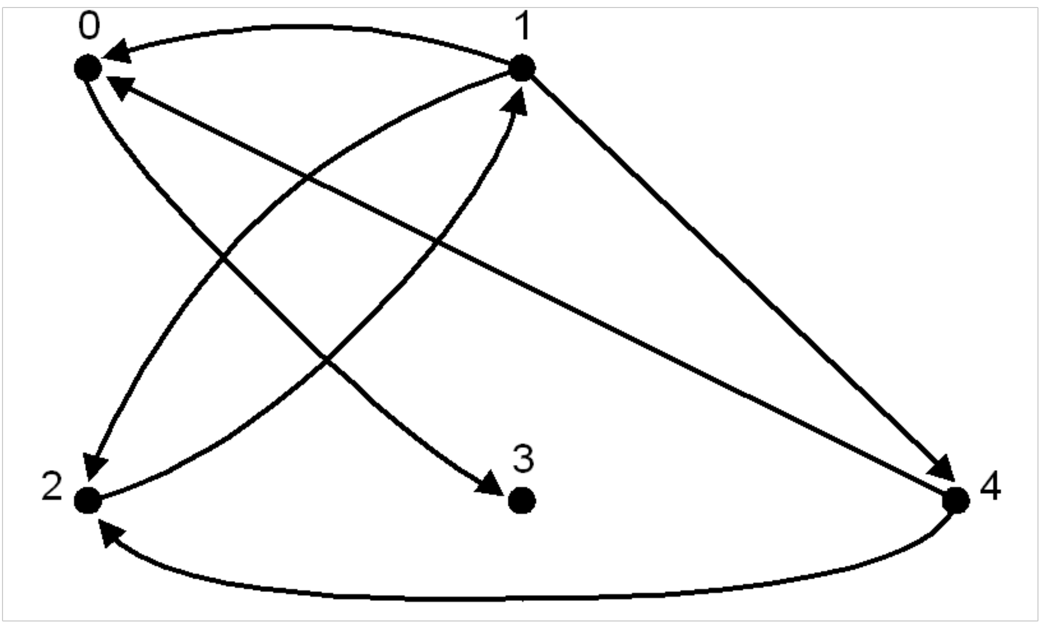
- A table with 1 row per vertex, listing its adjacent vertices.



Vertex	*Adjacent Vertices*
a	*b, c*
b	*a, c, e, f*
c	*a, b, f*
d	
e	*b*
f	*c, b*

Directed Adjacency Lists

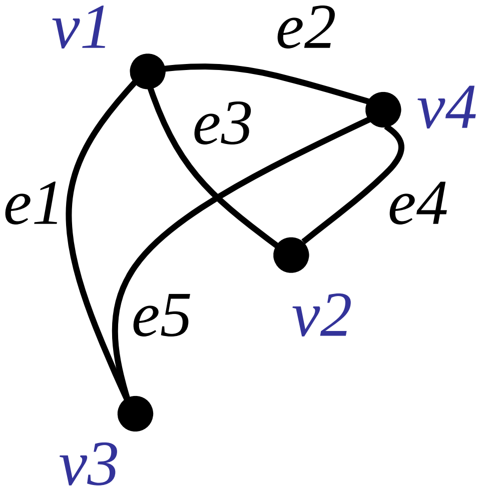
- 1 row per node, listing the terminal nodes of each edge incident from that node.



node	Terminal nodes
0	3
1	0, 2, 4
2	1
3	
4	0,2

Incidence matrices

- Matrix **M**=[m_{ij}], where m_{ij} is 1 when edge e_j is incident with v_i, 0 otherwise



	e1	e2	e3	e4	e5
v1	**1**	**1**	1	0	0
v2	0	0	**1**	**1**	0
v3	**1**	0	0	0	**1**
v4	0	1	0	1	1

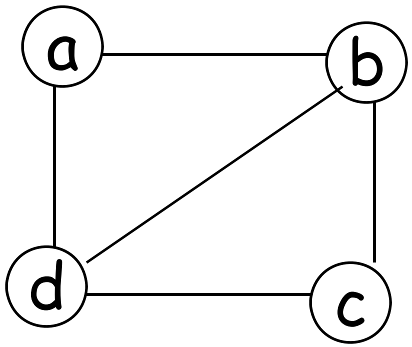
Isomorphism (Rosen 560 to 563)

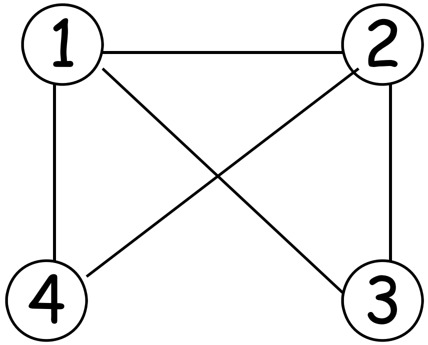
Are two graphs G1 and G2 of equal form?
- That is, could I rename the vertices of G1 such that the graph becomes G2
- Is there a bijection f from vertices in V1 to vertices in V2 such that
  - if (a,b) is in E1 then (f(a),f(b)) is in E2

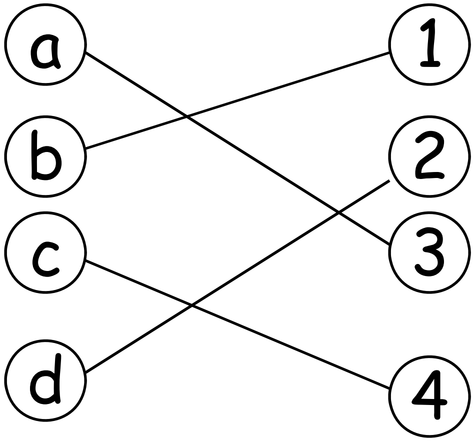
So far, best algorithm is exponential in the worst case

There are necessary conditions
- V1 and V2 must be same cardinality
- E1 and E2 must be same cardinality
- degree sequences must be equal
  - what's that then?

Are these graphs isomorphic?



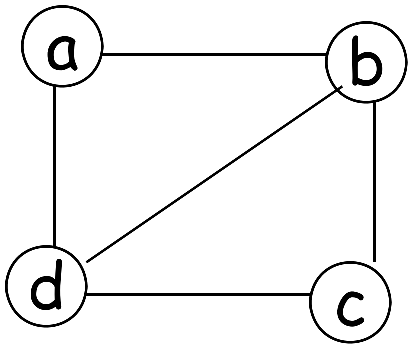


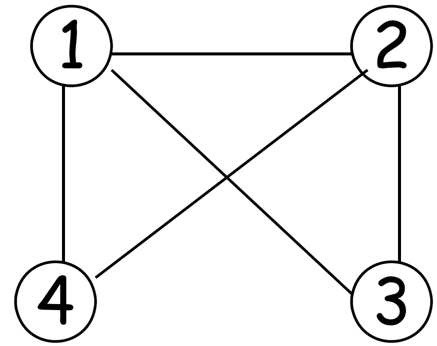


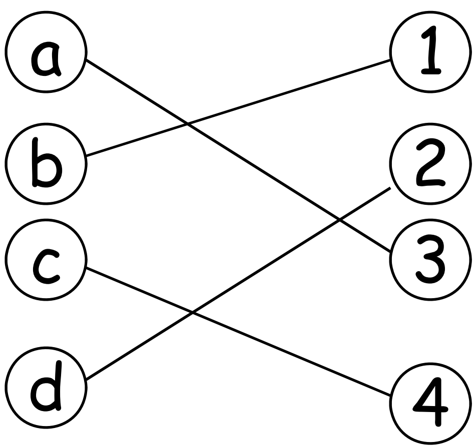
How many possible bijections are there?

Is this the worst case performance?

Are these graphs isomorphic?





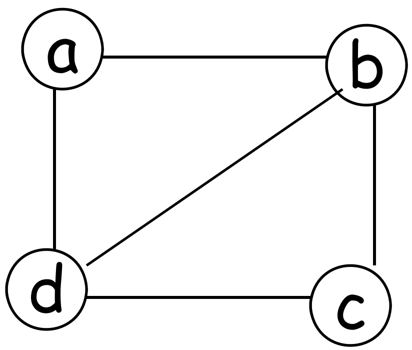


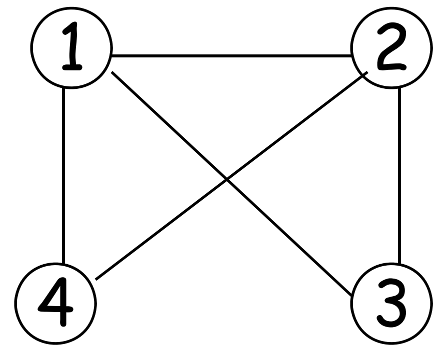
How many bijections?
1234,1243,1324,1342,1423,1432
2134,2143, …
…
4123,4132,… ,4321

4! = 4.3.2.1 = 24

Are these graphs isomorphic?

But not all 4! need be considered





What might the search process look like that constructs the bijection?

Connectivity

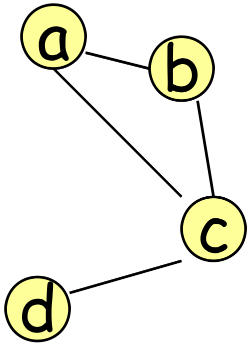
A Path of length n from v to u, is a sequence of edges that take us from u to v by traversing n edges.

A path is *simple* if no edge is repeated

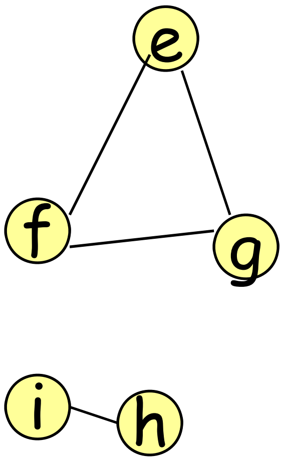
A *circuit* is a path that starts and ends on the same vertex

An undirected graph is connected if there is a path between every pair of distinct vertices

Connectivity



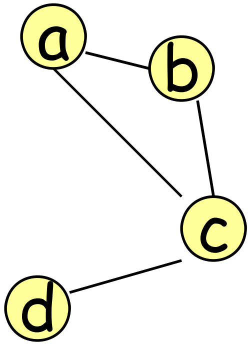
$$G_1 = (\{a,b,c,d\},\{(a,b),(a,c),(b,c),(c,d)\})$$



$$G_2 = (\{e,f,g,h,i\},\{(e,f),(e,g),(f,g),(i,h)\})$$

This graph has 2 components

Connectivity



c is a cut vertex
(d,c) is a cut edge

$$G = (\{a,b,c,d\},\{(a,b),(a,c),(b,c),(c,d)\})$$

A *cut vertex* v, is a vertex such that if we remove v, and all of the edges incident on v, the graph becomes disconnected

We also have a *cut edge*, whose removal disconnects the graph

Euler Path (the Konigsberg Bridge problem)

Rosen 8.5

Is it possible to start somewhere, cross all the bridges once only, and return to our starting place?

Leonhard Euler 1707-1783)

Is there a simple circuit in the given multigraph that contains every edge?

Fun with Paths in Graphs

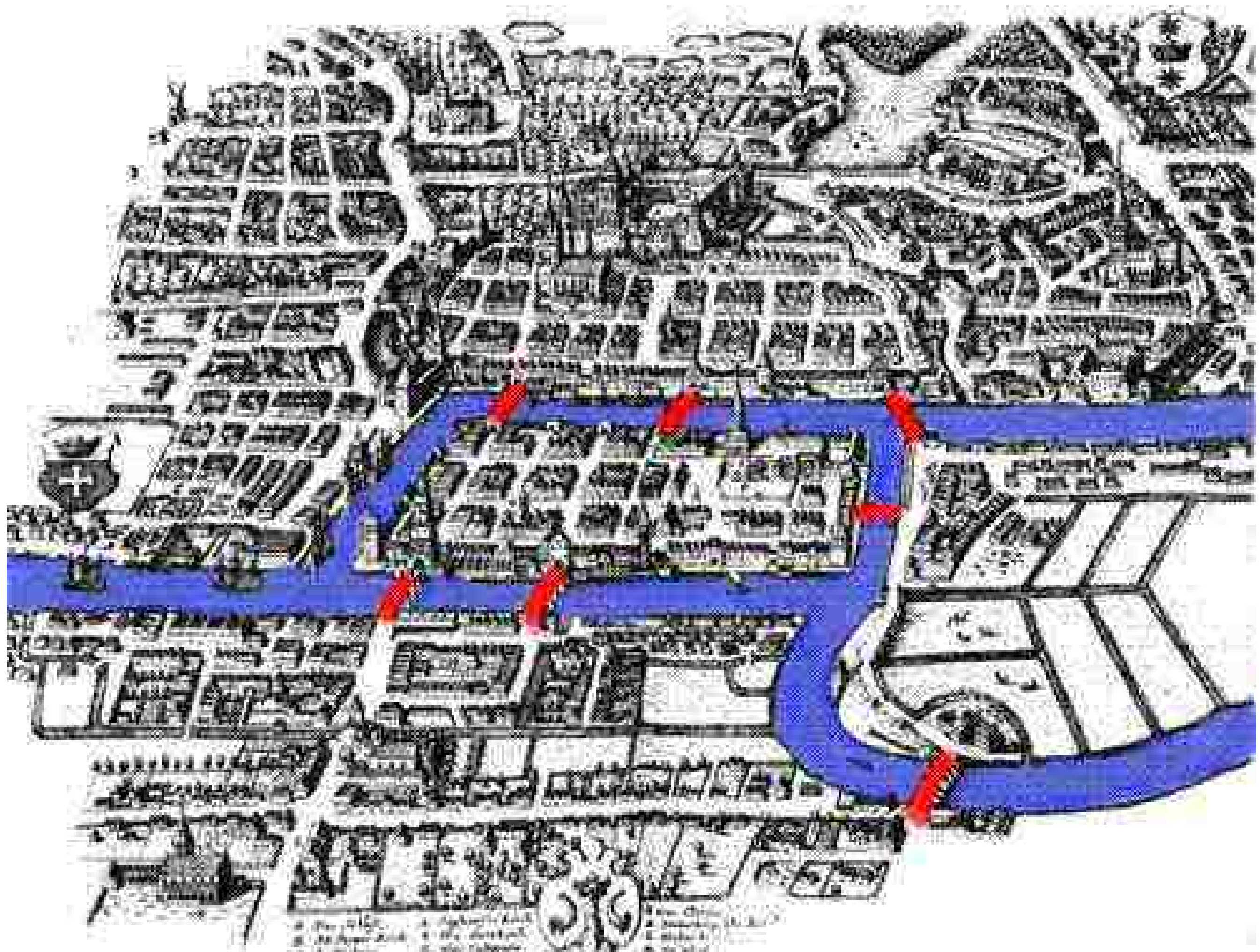
- "6 Degrees of Kevin Bacon Game"
  - Given a graph where:
    - V = { set of all actors and actresses }
    - E = { (a, b) | a,b $\in$ V and ($\exists$m $\in$Movies, a appeared in m and b appeared in m)}
  - Given an actor or actress A, can you find a path of length 6 or less from A to Kevin Bacon?

Slightly Less Fun Version (unless you're a mathematician)

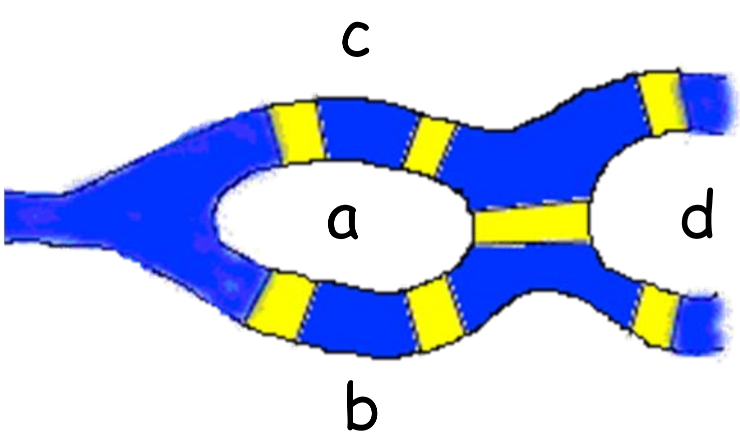
- The Erdos Number:
  - Given a graph where
    - V = { the set of all mathematicians and scientists from fields closely related to mathematics}
    - E = { (a,b) | a,b $\in$ V and "a coauthored an article, paper, or other scholarly work with b"}
  - Given a mathematician (or scientist from a field closely related to math) A, what's the length of the shortest path you can find from A to Paul Erdos?

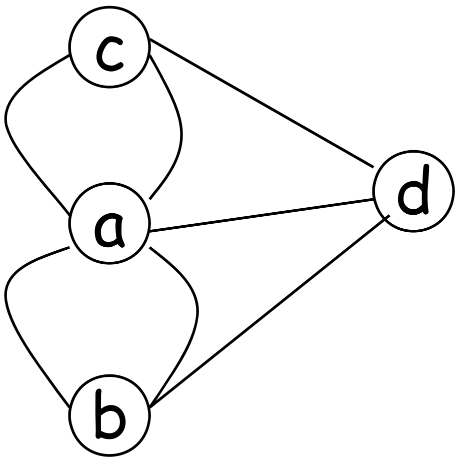
A few Erdos numbers

- A few famous scientists
  - Einstein: 2
  - Schrodinger: 8
  - John Nash: 4
- Another example (I was procrastinating one day while a grad student):
  - Cicirello: 4
  - Cicirello → Regli → Shokoufandeh → Szemerédi → Erdös



Euler Path (the Konigsberg Bridge problem)





Is there a simple circuit in the given multigraph that contains every edge?

An Euler circuit in a graph *G* is a simple circuit containing every edge of *G*.

An Euler path in a graph *G* is a simple path containing every edge of *G*.

Euler Circuit & Path

Necessary & Sufficient conditions

- every vertex must be of even degree
  - if you enter a vertex across a new edge
  - you must leave it across a new edge

A connected multigraph has an Euler circuit if and only if all vertices have even degree

The proof is in 2 parts (the biconditional)
The proof is in the book

Hamilton Paths & Circuits

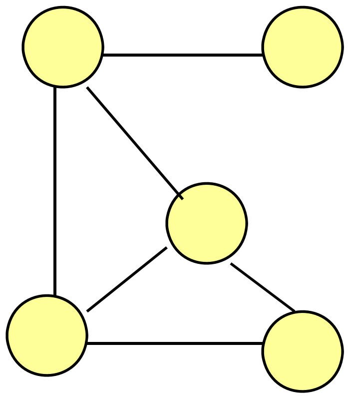
Given a graph, is there a circuit that passes through each vertex once and once only?

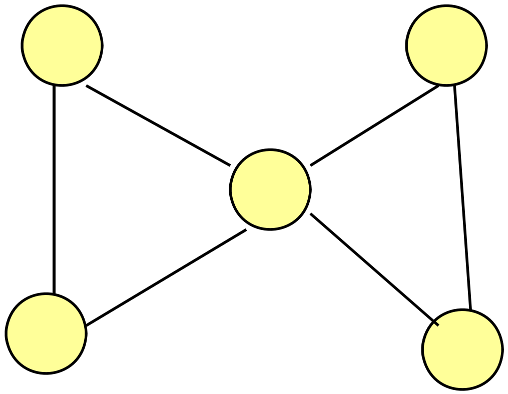
Given a graph, is there a path that passes through each vertex once and once only?

Easy or hard?

Due to Sir William Rowan Hamilton (1805 to 1865)

Hamilton Paths & Circuits





Is there an HC?

HC is an instance of TSP!

Connected?

Is the following graph connected?

$$G = (\{a, b, c, d, e, f, g\}, \{(a,b), (b,c), (b,d), (c,d), (g,e), (e,f), (f,g)\})$$

Draw the graph

What kind of algorithm could we use to test if connected?

Connected?

$$G = (\{a,b,c,d,e,f,g\},\{(a,b),(b,c),(b,d),(c,d),(g,e),(e,f),(f,g)\})$$

- (0) assume all vertices have an attribute visited(v)
- (1) have a stack S, and put on it any vertex v
- (2) remove a vertex v from the stack S
- (3) mark v as visited
- (4) let X be the set of vertices adjacent to v
- (5) for w in X do
  - (5.1) if w is unvisited, add w to the top of the stack S
- (6) if S is not empty go to (2)
- (7) the vertices that are marked as visited are connected