

# Relations and Functions: a review

# Relations

# Ordered n-tuples

- An ordered n-tuple is an ordered sequence of n objects
- $(x_1, x_2, \dots, x_n)$ 
  - First coordinate (or component) is  $x_1$
  - ...
  - n-th coordinate (or component) is  $x_n$
- An ordered pair: An ordered 2-tuple
  - $(x, y)$
- An ordered triple: an ordered 3-tuple
  - $(x, y, z)$

# Equality of tuples vs sets

- Two tuples are equal iff they are equal coordinate-wise
  - $(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$  iff  
 $x_1 = y_1, x_2 = y_2, \dots, x_n = y_n$
- $(2, 1) \neq (1, 2)$ , but  $\{2, 1\} = \{1, 2\}$
- $(1, 2, 1) \neq (2, 1)$ , but  $\{1, 2, 1\} = \{2, 1\}$
- $(1, 2-2, a) = (1, 0, a)$
- $(1, 2, 3) \neq (1, 2, 4)$  and  $\{1, 2, 3\} \neq \{1, 2, 4\}$

# Cartesian products

- Let  $A_1, A_2, \dots, A_n$  be sets
- The cartesian products of  $A_1, A_2, \dots, A_n$  is
  - $A_1 \times A_2 \times \dots \times A_n$   
 $= \{ (x_1, x_2, \dots, x_n) \mid x_1 \in A_1 \text{ and } x_2 \in A_2 \text{ and } \dots$   
 $\text{and } x_n \in A_n \}$
- Examples:  $A = \{x, y\}$ ,  $B = \{1, 2, 3\}$ ,  $C = \{a, b\}$
- $A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$
- $A \times B \times C = \{(x, 1, a), (x, 1, b), \dots, (y, 3, a), (y, 3, b)\}$
- $A \times (B \times C) = \{(x, (1, a)), (x, (1, b)), \dots, (y, (3, a)), (y, (3, b))\}$

# Relations

- A relation is a set of ordered pairs

- Let  $x R y$  mean  $x$  is  $R$ -related to  $y$
- Let  $A$  be a set containing all possible  $x$
- Let  $B$  be a set containing all possible  $y$

Relation  $R$  can be treated as a set of ordered pairs

$$R = \{(x, y) \in A \times B \mid x R y\}$$

- Example: We have the relation “is-capital-of” between cities and countries:

$$\text{Is-capital-of} = \{(\text{London, UK}), (\text{WashingtonDC, US}), \dots\}$$

# Relations are sets

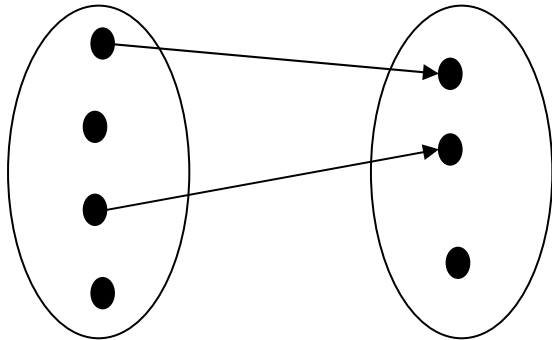
- $R \subseteq A \times B$  as a relation from  $A$  to  $B$
- $R$  is a relation from  $A$  to  $B$  iff  $R \subseteq A \times B$ 
  - Furthermore,  $x R y$  iff  $(x, y) \in R$ .
- If the relation  $R$  only involves two sets, we say it is a **binary relation**.
- We can also have an  $n$ -ary relation, which involves  $n$  sets.

# Various kinds of binary relations

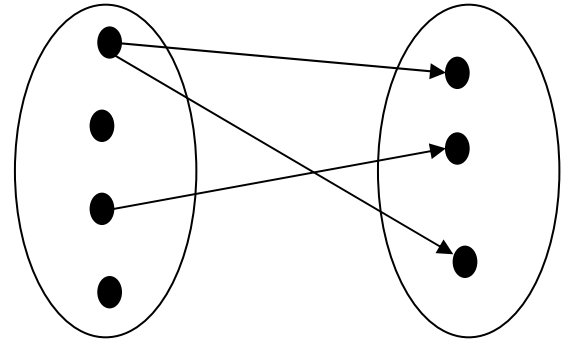
- **One-to-one relation**: each first component and each second component appear only once in the relation.
- **One-to-many relation**: if some first component  $s_1$  appear more than once.
- **Many-to-one relation**: if some second component  $s_2$  is paired with more than one first component.
- **Many-to-many relation**: if at least one  $s_1$  is paired with more than one second component and at least one  $s_2$  is paired with more than one first component.



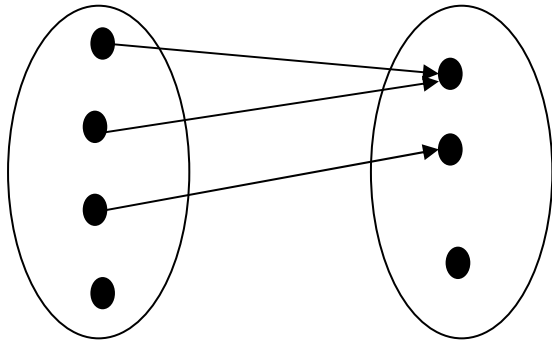
# Visualizing the relations



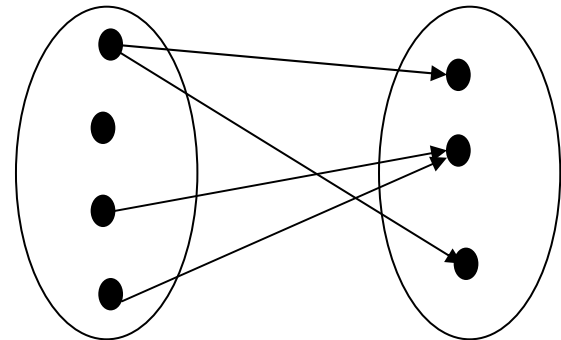
One-to-one



One-to-many



Many-to-one



Many-to-many

# Binary relation on a set

- Given a set  $A$ , a binary relation  $R$  on  $A$  is a subset of  $A \times A$  ( $R \subseteq A \times A$ ).
- An example:
  - $A = \{1, 2\}$ . Then  $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ .  
Let  $R$  on  $A$  be given by  $x R y \leftrightarrow x + y$  is odd.
  - then,  $(1, 2) \in R$ , and  $(2, 1) \in R$

# Properties of Relations: Reflexive

- Let  $R$  be a binary relation on a set  $A$ .
  - $R$  is reflexive: iff for all  $x \in A$ ,  $(x, x) \in R$ .
- Reflexive means that every member is related to itself.
- Example: Let  $A = \{2, 4, a, b\}$ 
  - $R = \{(2, 2), (4, 4), (a, a), (b, b)\}$
  - $S = \{(2, b), (2, 2), (4, 4), (a, a), (2, a), (b, b)\}$
- $R, S$  are reflexive relations on  $A$ .
- Another example: the relation  $\leq$  is reflexive on the set  $Z_+$ .

# Symmetric relations

- A relation  $R$  on a set  $A$  is symmetric iff for all  $x, y \in A$ , if  $(x, y) \in R$  then  $(y, x) \in R$ .
- Example:  $A = \{1, 2, b\}$ 
  - $R = \{(1, 1), (b, b)\}$
  - $S = \{(1, 2)\}$
  - $T = \{(2, b), (b, 2), (1, 1)\}$
- $R, T$  are symmetric relations on  $A$ .
- $S$  is not a symmetric relation on  $A$ .
- The relation  $\leq$  is reflexive on the set  $Z_+$ , but not symmetric. E.g.,  $3 \leq 4$  is in, but not  $4 \leq 3$

# Anti-symmetric relations

- A relation  $R$  on a set  $A$  is anti-symmetric iff for all  $x, y \in A$ . if  $(x, y) \in R$  and  $(y, x) \in R$  then  $x = y$ .
- Example:  $A = \{1, 2, b\}$ 
  - $R = \{(1, 1), (b, b)\}$
  - $S = \{(1, 2)\}$
  - $T = \{(2, b), (b, 2), (1, 1)\}$
- $R, S$  are anti-symmetric relations on  $A$ .
- $T$  is not an anti-symmetric relation on  $A$ .
- The relation  $\leq$  is reflexive on the set  $Z_+$ , but not symmetric. It is anti-symmetric.

# Transitive relations

- A relation  $R$  on a set  $A$  is transitive iff for all  $x, y, z \in A$ , if  $(x, y) \in R$  and  $(y, z) \in R$ , then  $(x, z) \in R$ .
- Example:  $A = \{1, 2, b\}$ 
  - $R = \{(1, 1), (b, b)\}$
  - $S = \{(1, 2), (2, b), (1, b)\}$
  - $T = \{(2, b), (b, 2), (1, 1)\}$
- $R, S$  are transitive relations on  $A$ .
- $T$  is not a transitive relation on  $A$ .
- The relation  $\leq$  is reflexive on the set  $\mathbb{Z}_+$ , but not symmetric. It is also anti-symmetric, and transitive (why?).

# Transitive closure

- Let  $R$  be a relation on  $A$
- The smallest transitive relation on  $A$  that includes  $R$  is called the transitive closure of  $R$ .
- Example:  $A = \{1, 2, b\}$ 
  - $R = \{(1, 1), (b, b)\}$
  - $S = \{(1, 2), (2, b), (1, b)\}$
  - $T = \{(2, b), (b, 2), (1, 1)\}$
- The transitive closures of  $R$  and  $S$  are themselves
- The transitive closure of  $T$  is  $T \cup \{(2, 2), (b, b)\}$

# Equivalence relations

- A relation on a set  $A$  is an equivalence relation if it is
  - Reflexive.
  - Symmetric
  - Transitive.
- Examples of equivalence relations
  - On any set  $S$ ,  $x R y \leftrightarrow x = y$
  - On integers  $\geq 0$ ,  $x R y \leftrightarrow x+y$  is even
  - On the set of lines in the plane,  $x R y \leftrightarrow x$  is parallel to  $y$ .
  - On  $\{0, 1\}$ ,  $x R y \leftrightarrow x = y^2$
  - On  $\{1, 2, 3\}$ ,  $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ .



# Functions

# Some example functions

- Functions are usually given by formulas
  - $f(x) = \sin(x)$
  - $f(x) = e^x$
  - $f(x) = x^3$
  - $f(x) = \log x$
- A function is a computation rule that changes one value to another value
- Effectively, a function associates, or relates, one value to another value.

# “general” functions

- We can think of a function as relating one object to another (need not be numbers).
- A relation  $f$  from  $A$  to  $B$  is a **function** from  $A$  to  $B$  iff
  - for every  $x \in A$ , there exists a unique  $y \in B$  such that  $x f y$ , or equivalently  $(x, y) \in f$
- Functions are also known as transformations, maps, and mappings.

# Notational convention

- Sometimes functions are given by stating the rule of transformation, for example,

$$- f(x) = x + 1$$

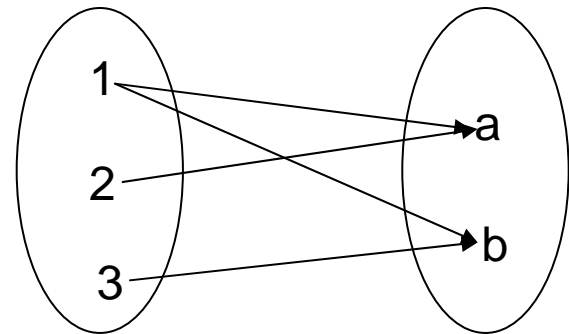
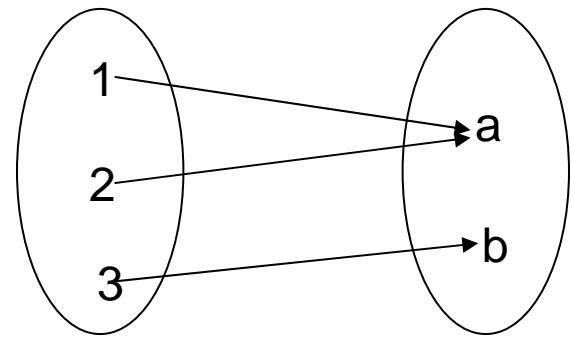
- This should be taken to mean

$$f = \{(x, f(x)) \in A \times B \mid x \in A\}$$

where  $A$  and  $B$  are some understood sets.

# Examples

- Let  $A = \{1, 2, 3\}$  and  $B = \{a, b\}$
- $R = \{(1, a), (2, a), (3, b)\}$  is **a function** from  $A$  to  $B$
- $R = \{(1, a), (1, b), (2, a), (3, b)\}$  is **not a function** from  $A$  to  $B$

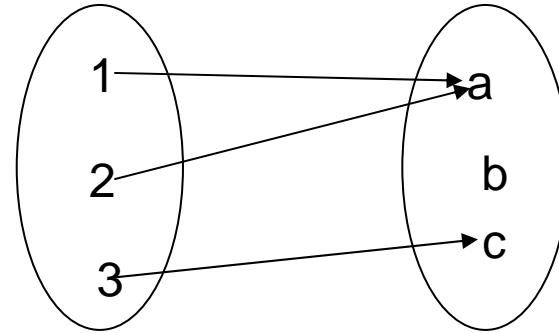


# Notations and concepts

- Let  $A$  and  $B$  be sets,  $f$  is a function from  $A$  to  $B$ . We denote the function by:  
 $f: A \rightarrow B$
- $A$  is the **domain**, and  $B$  is the **codomain** of the function.
- If  $(a, b) \in f$ , then  $b$  is denoted by  $f(a)$ ;  $b$  is the **image** of  $a$  under  $f$ ,  $a$  is a **preimage** of  $b$  under  $f$ .
- The range of  $f$  is the set of images of  $f$ .
  - The range of  $f$  is the set  $f(A)$ .

# An example

- Let the function  $f$  be



- Domain is  $\{1, 2, 3\}$
- Codomain is  $\{a, b, c\}$
- Range is  $\{a, c\}$

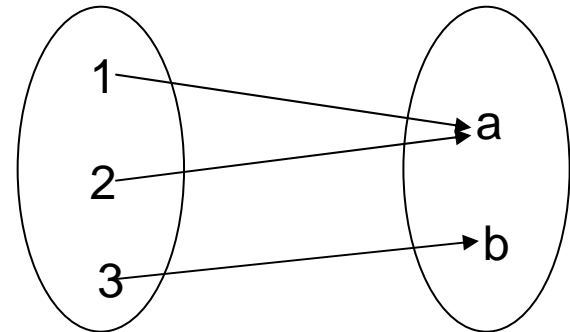
# Equality of functions

- Let  $f: A \rightarrow B$  and  $g: C \rightarrow D$ .
- We denote function  $f =$  function  $g$ 
  - iff set  $f =$  set  $g$
- Note that this force  $A = C$ , but not  $B = D$ 
  - Some require  $B = D$  as well.



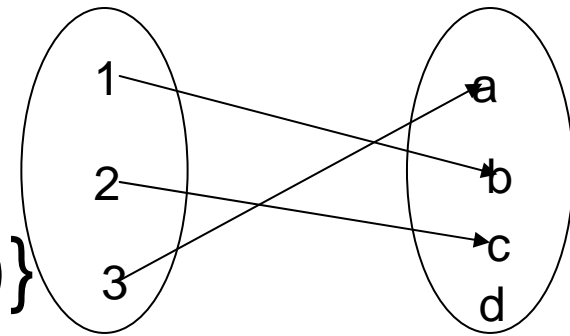
# Properties of functions: **onto**

- Let  $f: A \rightarrow B$ 
  - The function  $f$  is an **onto** or **surjective** function iff the range of  $f$  equals to the codomain of  $f$ .
  - Or for any  $y \in B$ , there exists some  $x \in A$ , such that  $f(x) = y$ .
- The function on the right is onto.
- $f: \mathbb{Z} \rightarrow \mathbb{Z}$  with  $f(x) = x^2$  is not onto



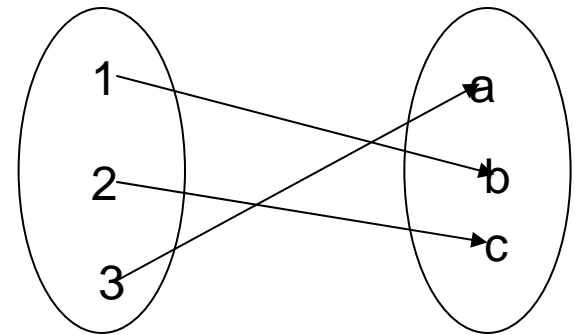
# One-to-one functions

- A function  $f: A \rightarrow B$  is **one-to-one**, or **injective** if no member of  $B$  is the image under  $f$  of two distinct elements of  $A$ .
- Let  $A = \{1, 2, 3\}$
- Let  $B = \{a, b, c, d\}$
- Let  $f = \{(1, b), (2, c), (3, a)\}$
- The function  $f$  is one-to-one
- $f: \mathbb{Z} \rightarrow \mathbb{Z}$  with  $f(x) = x^2$  is not one-to-one because  $f(2) = f(-2) = 4$ .



# Bijections (one-to-one correspondences)

- A function  $f: A \rightarrow B$  is **bijective** if  $f$  is both one-to-one and onto.
- Let  $A = \{1, 2, 3\}$
- Let  $B = \{a, b, c\}$
- Let  $f = \{(1, b), (2, c), (3, a)\}$
- The function  $f$  is one-to-one
- $f: \mathbb{Z} \rightarrow \mathbb{Z}$  with  $f(x) = x^2$  is not bijective because it is not one-to-one.



# Composition of functions

- Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$ . Then the composition function,  $g \circ f$ , is a function from  $A$  to  $C$  defined by  $(g \circ f)(a) = g(f(a))$ .
- Note that the function  $f$  is applied first and then  $g$ .
- Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^2$ .
- Let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $g(x) = \lfloor x \rfloor$ .  
$$(g \circ f)(2.3) = g(f(2.3)) = g((2.3)^2) = g(5.29)$$
$$= \lfloor 5.29 \rfloor = 5.$$

# Inverse functions

- **Identity function:** the function that maps each element of a set  $A$  to itself, denoted by  $i_A$ . We have  $i_A: A \rightarrow A$ .
- Let  $f: A \rightarrow B$ . If there exists a function  $g: B \rightarrow A$  such that  $g \circ f = i_a$  and  $f \circ g = i_b$ , then  $g$  is called **the inverse function** of  $f$ , denoted by  $f^{-1}$
- **Theorem:** Let  $f: A \rightarrow B$ .  $f$  is a bijection iff  $f^{-1}$  exists.
- **Example:**
  - $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = 3x+4$ .  $f^{-1} = (x - 4)/3$
  - $(f \circ f^{-1})(x) = 3(x-4)/3 + 4 = x$  identity function

# Summary

- We have introduced many concepts,
  - Function
  - Domain, codomain
  - Image, preimage
  - Range
  - Onto (surjective)
  - One-to-one (injective)
  - Bijection (one-to-one correspondence)
  - Function composition
  - Identity function
  - Inverse function