Combinatorial Optimization

What is “Combinatorial Optimization”?

First, what is “optimization”?

- **Informally:**
  - An optimization problem is a problem where we must find the “best” configuration of input parameters to achieve some goal.
- **Mathematically:**
  - Given a cost function: $c : F \rightarrow R$
  - F is any set
  - Find the $f \in F$, such that $c(f) \leq c(y)$ for all $y \in F$.

Optimization Problems

- An example
- Minimize $f : R \to R$, where $f(x) = 3\,x^2 + 10$

Combinatorial Optimization?

- What is Combinatorial Optimization?
- Two categories of optimization problems:
  - Those with continuous variables (example on previous slide)
  - Those with discrete variables
- Combinatorial Optimization (a.k.a. discrete optimization)

Combinatorial Optimization

- The domain of our cost function is either:
  - A finite set
  - Countably infinite set

Countably Infinite Sets

- A countably infinite set is a set whose elements can be specified through a one-to-one correspondence with the natural numbers or integers (and which has an infinite number of members)
- Examples:
  - The integers, the even integers, the odd integers, all multiples of the integer 3, etc

Countably Infinite Sets

- The following are not countably infinite:
  - The real numbers
  - The complex numbers
  - The irrational numbers

Combinatorial Optimization

- Definition:
  - Given a cost function: $c : F \rightarrow R$
  - F is any finite set or countably infinite set
  - Find the $f \in F$, such that $c(f) \leq c(y)$ for all $y \in F$
- Some common F:
  - The integers
  - Permutations of elements from some finite set S
  - Subsets of some finite set S
  - Graphs (will see what these are later in semester)
  - Trees (will see what these are later in semester)

An Example: A problem known as Bin Packing

- First a review of another set theory concept

Disjoint Sets

- Subsets of S, $s \subseteq S$, $t \subseteq S$, are disjoint subsets if:

$$s \cap t = \varnothing$$

A Partition of a Set

- A partition of a set S is a set P of subsets of S, such that $S = \bigcup_{p \in P}$ and $\forall_{p_1, p_2 \in P}\, p_1 \cap p_2 = \varnothing$

Bin Packing

- Given:
  - a set of objects: S
  - A function: $f : S \rightarrow Z_+$
  - A "bin size", M
- Find the partition P of S that minimizes |P|, such that, for all $p \in P$, $\sum_{e \in p} f(e) \leq M$

Bin Packing (informal description)

- You have a set of initially empty bins (e.g., boxes, containers, etc).
- Each bin can hold M units (M might be weight or some other measure of size)
- You have a set of objects (the set S)
- Each object has a size (the function f)
- Put the objects in bins without violating the weight limits M, while using the smallest number of bins